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Field Analysis and Potential Theory
Part 3

R.S. Edgar

DEPARTMENT OF DEFENCE
DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

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Electronics and Surveillance Research Laboratory

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ABSTRACT

Research Report

An aether-based treatment of space and time measurement is employed to investigate the rate of a moving clock, to develop doppler formulae, and to establish intra and inter-frame relationships in electrodynamics.

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NOTE ON ASSOCIATED PUBLICATIONS

Part 1 of this work was published in 1985 by the Commonwealth of Australia as Special Document ERL-0340-SD, under the title "Field Analysis and Potential Theory" (ISBN 0 642 11996 1).

It was republished in 1989 by Springer-Verlag, Heidelberg & New York, under the above title as No. 44 in the series "Lecture Notes in Engineering." (ISBN 3 540 51074 5 / 0 387 51074 5).

Part 2 was published in 1989 by the Commonwealth of Australia as Special Document ERL-0340-SD, subsequently amended to ERL-0506-SD, under the title "Field Analysis and Potential Theory, Part 2." (ISBN 0 642 15286 1).

EXECUTIVE SUMMARY

The significance of the Poincaré - Einstein procedure for clock synchronization is examined in the light of an aether - based treatment of space and time measurement, as developed by Ives. The analysis leads directly to the Lorentz transformations for inertial interframe measurements, and permits of a determination of the rate of a moving clock and the derivation of doppler formulae.

The interframe transformations for the electromagnetic density and field functions are developed in considerable generality, and Lorentz-covariance of the Maxwell-Lorentz equations demonstrated.

Given a self-contained source complex which moves as a whole, the measured values of the electromagnetic functions in a neighbourhood of a particular source element are shown to depend only upon the velocity of the complex, and to be independent of the frame of reference in which the measurements take place. As a consequence, the constitutive equations, when modified for motion within the aether frame, continue to hold in all inertial frames.

The conventional relativistic treatment of Maxwell's equations and allied topics is criticised on the grounds that it is essentially asymmetrical and leads, *inter alia,* to the belief that only relative velocity is significant in electromagnetic interactions.

PREFACE

Although Einstein's Special (or Restricted) Theory of Relativity was formulated nearly ninety years ago, it has never been accepted by the whole of the physics community, let alone the community at large; it has, indeed, been the subject of attack at regular intervals. This is deplored by the relativist, Paul Davies, who has written:[1]

> "..... the scale and ferocity of the attack on the theory of relativity indicates a deep-rooted cultural antipathy. Undoubtedly many feel resentful about their cherished intuition being upset. Dealing as it does with fundamental concepts such as space and time, relativity is bound to produce a hostile and sceptical public reaction. There is also an element of simple confusion that encourages the repeated resurrection of the same old 'paradoxes' and controversies with wearisome persistence. ..... By making thought experiments sufficiently complicated a wily anti-relativist can frequently bury an error amid a maze of observers, clocks and rockets whizzing in various directions. ..... Ultimately all these clever attempts to find a loophole in special relativity are doomed to failure on logical grounds."

On the other hand, anti-relativists voice a similar complaint:[2]

> "..... there are inherent difficulties in arguing with relativists. Because of the abstractedness, the lack of definition and elaboration of its terms and the ambiguous and inconsistent use of them, relativity has endowed itself with considerable conceptual elasticity. ..... Arguments invariably bog down because of difficulties in communication. Relativity always appears to be misunderstood by the critic. ..... Whenever a relativist refers to a 'standard concept' it is very unlikely that the concept has an agreed meaning or even a confluent meaning for all participants in a discussion."

And again:[3]

> "..... Before a relativist's mathematical operations get under way, his challenger should insist on rigorous *physical* definitions of the mathematical symbols; moreover he should see to it that the meanings do not shift in the course of the discussion. ..... A watch may be kept for the relativist's use of a concept as definite which he has earlier denied as having a precise or general meaning. The relativist may be asked whether his procedure is generally accepted by relativists or only by some of them. ..... The challenger should not

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1 New Scientist, 7 August, 1980, p 463.

2 N. Rudakov, "Fiction Stranger than Truth." The Author, Geelong, 1981.

3 D. Turner and R. Hazelett, "The Einstein Myth and the Ives Papers" Pt. 2. Devin-Adair Co. Old Greenwich, Connecticut, 1979.

be embarrassed by his own insistence in asking these plodding questions in the face of the relativist's customary facility in the manipulation of mathematics."

One may well enquire how such a situation, having once arisen, could have persisted for so long. The simple answer is that Einsteinian relativity does violence to our intuitive concepts - in particular, to that of distant simultaneity - and in so doing places severe constraints upon the manner in which arguments involving space and time may be carried on. This is a factor which is not recognised by the non-relativist, who insists upon applying a commonsense approach to a subject which, in conventional exposition, does not lend itself to such treatment.

Thus, in the celebrated exchange between Dingle and McCrea some thirty years ago[4], when Dingle claimed to have discovered an internal inconsistency in the special theory, McCrea countered his assertion by stating that Dingle was in error because 'he deals with objects to which the theory denies a meaning .... about the first thing that relativity theory does is to deny any operational meaning to the notion of simultaneity at different places.'
As would be expected, this pronouncement failed to inform Dingle and his fellow critics.[5]

One should not anticipate an early end to the debate, given the long-standing arguments between relativists over the solution of such an apparently simple problem as the clock paradox, and in view of the cavalier disregard of their own rules which some relativists display in their published works.

But there is a way out of this real (or apparent) impasse; it is no less than a return to pre-Einsteinian physics - to the relativity of Poincaré and Lorentz. The case for this has been argued forcibly by Builder.[6]

> "The permissibility of retaining the concepts of absolute space, of absolute motion and of the ether, and the fact that we can assign to these concepts definite and clear meanings compatible with the restricted theory of relativity has striking pedagogical and heuristic advantages.
>
> The conceptual difficulties associated with the restricted theory all arise out of the denial that the absolute concepts are permissible and out of consequent attempts to avoid them in the presentation of the theory. It is frequently maintained that the theory has forced us to discard entirely the old-fashioned commonsense notions of time and space; but nothing comprehensible or definable has been offered in their place. Moreover, any questions as to what *causes* the relativity of simultaneity, the measured constancy of the velocity of light in all inertial reference systems or the reciprocity of relativistic variations of length, of mass and of clock rates are evaded by vague references to the principle of relativity, to the four-dimensional character of space-time, and so on.

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4 Nature; vol. 216, Oct.14, 1967; vol. 217, Jan. 6, 1968.
Dingle's argument involved the assumption that separated 'synchronized' clocks would show the same reading at the same (common-sense) instant.

5 G. Burniston-Brown, Inst. Phys. & Phys. Soc. Bulletin, 18 (1967) p. 73. This article brings Dingle's paradox into sharper focus.

6 G. Builder, "Ether and Relativity", Aust. J. Phys. 11 (1958) pp. 279-297.

On the other hand, the presentation of the theory in terms of the absolute concepts (following generally the lines of development by Poincaré and Lorentz) involves no conceptual difficulties. The relativity of simultaneity, the reciprocity of relativistic variations and the constancy of the measured velocity of light, then all appear simply as comprehensible effects of the motions relative to the ether of the bodies observed and of the measuring instruments used.

..... The heuristic value of this approach is also noteworthy. It reduces many questions which would otherwise lead to discursive and inconclusive arguments to a form in which a simple and conclusive answer can be given. For example, the relative retardation of clocks predicted by the restricted theory becomes a simple and intelligible consequence of the motion of the clocks relative to the ether.

It is worth remarking that the pedagogical and heuristic advantages of this approach depend only on the tenability of the ether hypothesis and on the admissibility of the absolute concepts, i.e. on their compatibility with the restricted theory and with the general body of physical knowledge. These advantages would remain even if there were also available an alternative and equally tenable set of hypotheses and concepts."

What, then, is the framework within which we may carry out the program suggested above?

Firstly, it is postulated that a frame of reference (aether) exists in which the speed of light is c in all directions. Clocks which are stationary in the aether and have been set by the Poincaré-Einstein out-and-back light synchronization procedure will consequently show the same reading at a given instant.[7]

Secondly, three fundamental assumptions are made relating to motion with velocity $\bar{u}$ in the aether frame, viz. that

(1) rods contract in the direction of motion by the factor $\left(1-u^2/c^2\right)^{1/2}$

(2) clocks are slowed by the factor $\left(1-u^2/c^2\right)^{1/2}$

(3) the momentum of a particle of mass m takes the form $m\bar{u}\big/\left(1-u^2/c^2\right)^{1/2}$

Rod contraction was first suggested by Fitzgerald in 1889 as a possible explanation of the null result of the Michelson-Morley experiment, on the assumption that the aether was not earth-convected; in 1900 Larmor pointed out that clock slowing would be a necessary accompaniment of rod contraction if the out-and-back speed of light were the same in different inertial frames of reference. These two effects, in combination, are sufficient to account for the null result of the Kennedy-Thorndike experiment of 1932 in which the Michelson-Morley experiment was repeated with unequal interferometer arms.

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[7] cf. Lorentz: "My notion of time is so definite that I clearly distinguish in my picture what is simultaneous and what is not."

The slowing of 'atomic' clocks by the factor $\left(1-u^2/c^2\right)^{1/2}$ was experimentally confirmed by Ives and Stilwell in 1938[8]; the momentum relationship has been the subject of experimental demonstration for high-speed electrons over a period of years[9].

Given assumptions (1) and (2) it is possible to determine the rate of a moving clock in any inertial frame, to develop the corresponding doppler formulae and to derive the Lorentz transformations. In addition, the analysis reveals that clocks in a non-aetherial inertial frame which have been synchronized by the Poincaré - Einstein procedure will not, in general, display a common reading at any instant[10].
These matters are dealt with in Chapter 1 below.

In Chapter 2 the Lorentz transformations are applied to the parameters of the electrical model which comprises statistically-regular configurations of singlets and of doublets and whirls in uniform translation in the aether frame, in order to determine the interframe transformations of the density functions $\rho, \overline{J}, \overline{P}$ and $\overline{M}$. The $\overline{E}$ and $\overline{B}$ transformations follow from the transformations for the retarded potentials, since $\overline{E}$ and $\overline{B}$ are defined in terms of them. The latter transformations, together with those for the density functions, are substituted in the Maxwell-Lorentz equations for a moving medium as developed in an earlier work[11], and Lorentz-covariance of the equations demonstrated.

Assumption (3) is then utilised to derive transformations for mechanical and electrical force between inertial frames, given that the Lorentz force formula holds within the aether frame.

Whereas our primary concern in Chapter 2 has been the relationship between measurements carried out in two inertial frames upon a given source complex (having a particular absolute velocity), we proceed in Chapter 3 to investigate the manner in which the measured values of the source parameters change when a source complex, which moves as a whole, assumes different velocities within a given frame of reference (and thereby acquires different absolute velocities). In the latter case we are concerned with real, physical changes in source configuration rather than with accidents of measurement. The computed changes in the dimensions of the electrical complex and in the frequency of its cyclic component are, in fact, found to be in accord with the variations expressed in (1) and (2) above for material rods and clocks, provided that the system is subject to Lorentz forces alone (or their mechanical equivalents).

It is further shown that for a given set of source elements which translate as a whole, the measured values of the density functions and of $\overline{E}$ and $\overline{B}$ will be equal at corresponding points in all inertial frames for the same measured value of velocity. This permits of the extrapolation to all inertial frames of the modified constitutive equations, as developed for a moving medium within the aether frame.

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8 In such experiments $\overline{u}$ is, ofcourse, velocity as measured (or calculated) in the earth frame. It will be shown subsequently that the factor $\left(1-u^2/c^2\right)^{1/2}$ carries over from the aether frame into all inertial frames of reference.

9 For an historical review, see Miller: "Albert Einstein's Special Theory of Relativity" Sec. 12.4. Addison-Wesley, 1981.

10 This resolves Dingle's paradox.

11 "Field Analysis and Potential Theory" Pt. 2. (The initials F.A.P.T. will be used to denote this work in subsequent foot-notes.)

Chapter 4 is a critique of the conventional approach to Maxwell's equations for a moving medium. It is claimed that the Minkowski treatment is intrinsically asymmetrical, and inferior to the Lorentz form in which medium velocity appears explicitly. Moreover, the ascription of whirl moment to a translating doublet, as required in the retrospective justification of a particular transformation, is clearly in error.

[Postscript: Subsequent to the completion of the present work, an interesting paper by H. Erlichson[12] has come to the attention of the writer. This contains an informative review of the history of the subject under discussion and contains an extensive set of references. Erlichson attributes the first appreciation of the physical nature of clock retardation to Einstein rather than Larmor; the concept was later adopted by Lorentz.

Erlichson could, perhaps, have laid greater stress on the fundamental difference between clock retardation as envisaged by Einstein and as understood in the 'absolute' theory, and have given greater prominence to the conceptual difficulties of the special theory, as recorded by Builder.]

R.S. Edgar

12 "The Rod Contraction - Clock Retardation Ether Theory and the Special Theory of Relativity." Am. J. Phys. 41 (1973) pp.1068-1077.

NOTATION

As in Parts 1 & 2,

(1) vector quantities are represented by a bar over the associated symbol.

(2) Gaussian units are employed throughout.

Equation numbering duplicates that in Parts 1 & 2 but cross references refer only to Part 3.

CHAPTER 1

AN AETHER-BASED TREATMENT OF SPACE AND TIME MEASUREMENT

1.1 Poincaré-Einstein Clock Synchronization and the Speed of Light [1]

We proceed in this section to show that if rods and clocks are subject to the Fitzgerald-Larmor space and time contractions when moving through the aether, the two-way speed of light will be measured as c in all inertial frames of reference, and in any non-aetherial inertial frame the Poincaré-Einstein clock synchronization procedure will result in continuous epoch retardation in the direction of absolute motion.

Suppose that a plane platform moves with velocity $\overline{W}$ with respect to the aether frame. Let $\overline{W}$ define the direction of the positive x axis and let the platform define the xy plane. (Fig. 1.1)

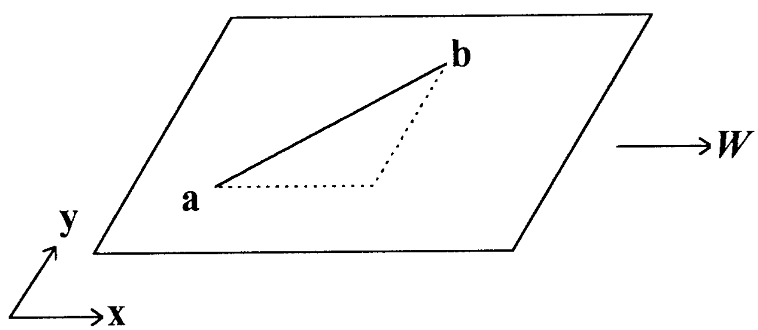

Fig. 1.1

A clock moves from **a** to **b** with uniform speed q relative to the platform, as measured by the clock itself in passing over platform graduations. Its speed relative to the platform, as measured in the aether frame, is Y. The transit time shown on the moving clock is τ' while that shown on any clock stationary in the aether frame is τ.

[1] The analysis in this section is based, in large part, upon that developed by Ives and employs the same symbols.
H.E. Ives "The Measurement of the Velocity of Light by Signals Sent in One Direction." J. Opt. Soc. Am. 38 (1948) pp. 879-884.
"Lorentz-Type Transformations as Derived from Performable Rod and Clock Operations." J. Opt. Soc. Am. 39 (1949) pp. 757-8 only.

The length **ab** is measured on the platform as R' and as R in the aether frame. The x, y components of R' are D', y, while those of R are D, y, (since there is no compression in the y direction). The angle made by $\overrightarrow{\mathbf{ab}}$ with the positive x axis is measured as θ' on the platform and as θ in the aether frame.

Since the x dimensions of the platform and those of rulers employed on the platform contract equally when in motion, D' will represent the x component of **ab** when the platform is at rest. Hence, when the platform is in motion,

$$D = D'\left(1 - W^2/c^2\right)^{1/2} \tag{1.1-1}$$

and $$R^2 = D^2 + y^2 = R'^2\left\{1 - \frac{W^2}{c^2}\frac{D'^2}{R'^2}\right\}$$

or $$R = R'\left(1 - W^2\cos^2\theta'/c^2\right)^{1/2} \tag{1.1-2}$$

Also $$\cos\theta = D/R = \frac{\cos\theta'\left(1 - W^2/c^2\right)^{1/2}}{\left(1 - W^2\cos^2\theta'/c^2\right)^{1/2}} \tag{1.1-3}$$

The speed of the moving clock relative to the aether frame is

$$\left\{(Y\cos\theta + W)^2 + (Y\sin\theta)^2\right\}^{1/2} = \left(Y^2 + W^2 + 2WY\cos\theta\right)^{1/2}$$

hence

$$\tau' = \tau\left\{1 - \frac{Y^2 + W^2 + 2WY\cos\theta}{c^2}\right\}^{1/2} \tag{1.1-4}$$

But $$Y = \frac{R}{\tau} = \frac{R'}{\tau'}\left\{1 - \frac{W^2}{c^2}\cos^2\theta'\right\}^{1/2}\left\{1 - \frac{Y^2 + W^2 + 2WY\cos\theta}{c^2}\right\}^{1/2}$$

On squaring both sides, collecting terms, writing $R'/\tau' = q$ and substituting for $\cos\theta$ from (1.1-3), we obtain the quadratic equation

$$Y^2\left\{1 + \frac{q^2}{c^2}\left(1 - \frac{W^2}{c^2}\cos^2\theta'\right)\right\} + Y\frac{2Wq^2}{c^2}\left(1 - \frac{W^2}{c^2}\right)^{1/2}\cos\theta'\left(1 - \frac{W^2}{c^2}\cos^2\theta'\right)^{1/2}$$
$$- \left(1 - \frac{W^2}{c^2}\right)q^2\left(1 - \frac{W^2}{c^2}\cos^2\theta'\right) = 0$$

of which the solution is

$$Y = \frac{q\left(1-W^2/c^2\right)^{1/2}\left(1-W^2\cos^2\theta'/c^2\right)^{1/2}}{\left(1+\dfrac{q^2}{c^2}\right)^{1/2}+\dfrac{qW}{c^2}\cos\theta'} \tag{1.1-5}$$

Substitution of (1.1-5) in (1.1-4) then yields

$$\tau' = \frac{\tau\left(1-W^2/c^2\right)^{1/2}}{\left(1+\dfrac{q^2}{c^2}\right)^{1/2}+\dfrac{qW}{c^2}\cos\theta'} \tag{1.1-6}$$

Since the elapsed time of the clock movement as measured in the aether frame is τ, the corresponding elapsed time registered by a clock which remains fixed at the point **a** on the platform will be $\tau\left(1-W^2/c^2\right)^{1/2}$, so that when the moving clock arrives at **b** it will have fallen behind the clock at **a** by

$$\Delta = \tau\left(1-W^2/c^2\right)^{1/2} - \frac{\tau\left(1-W^2/c^2\right)^{1/2}}{\left(1+\dfrac{q^2}{c^2}\right)^{1/2}+\dfrac{qW}{c^2}\cos\theta'}$$

whence $$\Delta = \tau\left(1-W^2/c^2\right)^{1/2}\left\{\frac{\left(1+q^2/c^2\right)^{1/2}-1+qW\cos\theta'/c^2}{\left(1+q^2/c^2\right)^{1/2}+qW\cos\theta'/c^2}\right\}$$

or, from (1.1-6), $$\Delta = \frac{R'}{q}\left\{\left(1+\frac{q^2}{c^2}\right)^{1/2}-1+\frac{qW}{c^2}\cos\theta'\right\} \tag{1.1-7}$$

We are now in a position to determine the measured speed of a particle which moves from **a** to **b**, when its transit time is taken to be its arrival time, as shown on the clock which has moved to **b**, minus its departure time as shown on the clock fixed at **a.** (The two clocks, when stationary, are, of course, supposed to run at the same rate and are set to the same time before one moves to **b**)

If the transit time is t when measured by a clock fixed in the aether frame, the clocks at both **a** and **b** will change by $t\left(1-W^2/c^2\right)^{1/2}$ during the movement of the particle. But since the epoch of the clock at **b** is Δ behind that at **a**, the apparent transit time as defined above is given by

$$t' = t\left(1-W^2/c^2\right)^{1/2} - \Delta$$

or $$t = (t'+\Delta)\Big/\left(1-W^2/c^2\right)^{1/2} \tag{1.1-8}$$

If V is the speed of the particle relative to the platform as measured in the aether frame, we have

$$V = \frac{R}{t} = \frac{R'}{t}\left\{1 - \frac{W^2 \cos^2 \theta'}{c^2}\right\}^{1/2} = R' \frac{\left(1 - W^2 \cos^2 \theta'/c^2\right)^{1/2}\left(1 - W^2/c^2\right)^{1/2}}{t'\left(1 + \Delta/t'\right)}$$

i.e.

$$V = \frac{Q_2\left(1 - W^2 \cos^2 \theta'/c^2\right)^{1/2}\left(1 - W^2/c^2\right)^{1/2}}{1 + \dfrac{Q_2}{q}\left\{\left(1 + \dfrac{q^2}{c^2}\right)^{1/2} - 1 + \dfrac{qW}{c^2}\cos\theta'\right\}} \qquad (1.1\text{-}9)$$

where $Q_2 = R'/t'$ = two-clock speed as measured on the platform.

We now apply the above considerations to light transmission. Suppose that a light photon emitted from **a** is received at **b** after an elapsed time t as measured in the aether frame. If V is the speed of the photon relative to the platform, as measured in the aether frame, the appropriate vector relationship is shown in Fig. 1.2.

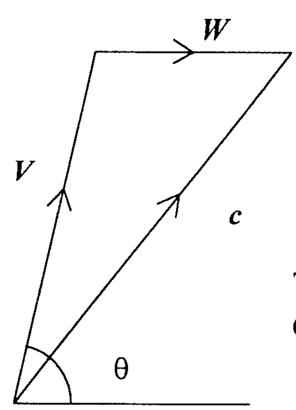

Fig. 1.2

Then $c^2 = W^2 + V^2 + 2VW\cos\theta$

On solving the quadratic for V we get

$$V = -W\cos\theta + (W^2\cos^2\theta + c^2 - W^2)^{1/2}$$

Substitution from (1.1-3) then yields

$$V = \frac{\left(1 - W^2/c^2\right)^{1/2}(c - W\cos\theta')}{\left(1 - W^2\cos^2\theta'/c^2\right)^{1/2}} \qquad (1.1\text{-}10)$$

On combining (1.1-9) and (1.1-10) we find that

$$Q_2 = \frac{c}{1 - \dfrac{c}{q}\left\{\left(1 + \dfrac{q^2}{c^2}\right)^{1/2} - 1\right\}} \qquad (1.1\text{-}11)$$

This is the speed of light as computed from platform measurements with two clocks. It is seen to be independent of the platform velocity relative to the aether frame but dependent upon the self-measured velocity of the moved clock.

It is of interest to consider what modifications of formulae may be required if particle movement is directed from **b** to **a**.

Since the epoch of clock **a** is ahead of that at **b**, (1.1-8) is replaced by

$$t = \frac{t' - \Delta}{\left(1 - W^2/c^2\right)^{1/2}} \tag{1.1-8a}$$

and (1.1-9) becomes

$$V = \frac{Q_2\left(1 - W^2\cos^2\theta'/c^2\right)^{1/2}\left(1 - W^2/c^2\right)^{1/2}}{1 - \dfrac{Q_2}{q}\left\{\left(1 + \dfrac{q^2}{c^2}\right)^{1/2} - 1 + \dfrac{qW}{c^2}\cos\theta'\right\}} \tag{1.1-9a}$$

If a photon is emitted from **b** and received at **a** the appropriate vector diagram is now that of Fig. 1.3.

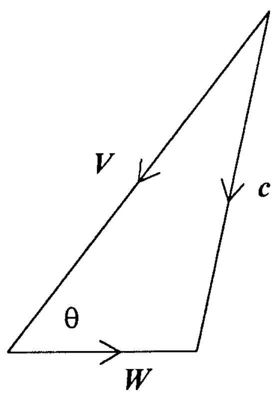

Fig. 1.3

In this case $c^2 = V^2 + W^2 - 2VW\cos\theta$

whence

$$V = W\cos\theta + (W^2\cos^2\theta + c^2 - W^2)^{1/2}$$

Substitution from (1.1-3) then yields

$$V = \frac{\left(1 - W^2/c^2\right)^{1/2}(c + W\cos\theta')}{\left(1 - W^2\cos^2\theta'/c^2\right)^{1/2}} \tag{1.1-10a}$$

and on combining (1.1-9a) and (1.1.-10a) we get

$$Q_2 = \frac{c}{1 + \dfrac{c}{q}\left\{\left(1 + \dfrac{q^2}{c^2}\right)^{1/2} - 1\right\}} \tag{1.1-11a}$$

(1.1-11) and (1.1-11a) differ by a sign in the denominator, hence the speed of light as measured on the platform, while being independent of the platform velocity relative to

the aether frame, is not the same in the directions **ab** and **ba**. Nevertheless, we may show that the two-way speed of light, as measured on the platform by a single clock, will always be c. Thus if a light photon leaves **a** at time t_1' on the **a** clock, reaches **b** at time t_2' on the **b** clock and, after reflection at **b**, returns to **a** at time t_3' on the **a** clock, then

$$t_2' - t_1' = R' \Bigg/ \frac{c}{\left[1 - \frac{c}{q}\left\{\left(1+\frac{q^2}{c^2}\right)^{1/2} - 1\right\}\right]} \quad \text{and} \quad t_3' - t_2' = R' \Bigg/ \frac{c}{\left[1 + \frac{c}{q}\left\{\left(1+\frac{q^2}{c^2}\right)^{1/2} - 1\right\}\right]}$$

whence, by addition,

$$t_3' - t_1' = 2R'/c$$

thus yielding a mean speed of c over the two-way path.

A considerable simplification is effected if we postulate $q \to 0$. Then (1.1-7) reduces to $\Delta = R' \cos\theta' W/c^2$, and, if the point **a** is identified with the origin of platform co-ordinates,

$$\Delta = W x'/c^2 \tag{1.1-12}$$

Since the epoch slip now depends only upon x', the mobile clock may move in an arbitrary path between **a** and **b**. In a multiple-clock situation it is not necessary that all clocks move out from the origin; any one clock may proceed to its final position from the neighbourhood of another clock whose epoch it initially shares. The each-way speed of light between **a** and **b** now reduces to c, as may be shown by putting $q \to 0$ in (1.1-11) and (1.1-11a). Furthermore, this will be the each-way speed as measured between any two points on the platform since the preceding analysis may be applied equally to these points, and the differential epoch slip between them has the appropriate value.

In these circumstances we can set any fixed clock **b** relative to a fixed clock **a** by arranging **b**'s reading to be the mean of the emission and reception times at **a** of a light-pulse reflected without delay at **b**. This eliminates the philosophical problem of distant clock setting by infinitely slow movement[2], and is identical with the Poincaré-Einstein procedure for 'synchronizing' spaced clocks. Clearly, it does not lead to true synchronization, but we will continue to employ the term to describe this type of operation.

It is seen, then, that clocks to the right of the origin, i.e. in the direction of $\overline{W}$, are retarded while those to the left are advanced, and those on the y axis share the origin epoch. Had the platform been moving towards the left, clocks to the left would have been delayed and those to the right advanced. In the latter case it may be shown that if

[2] It follows from (1.1-7) that the same result obtains when clocks are moved rectilinearly from the origin with finite velocity, and advanced at the completion of the movement by $\tau'\left\{\left(1+\frac{q^2}{c^2}\right)^{1/2} - 1\right\}$, both τ' and q being known.

$\overrightarrow{\mathbf{ab}}$ continues to make angles θ and θ' with the positive x axis, W must be replaced by $-W$ in (1.1-5, 6, 7, 9, 9a, 10, 10a).

So far, we have restricted considerations to a plane region (platform). We can cover exterior points as follows:

let $\mathbf{A}(x_1', y_1', z_1')$ and $\mathbf{B}(x_2', y_2', z_2')$ be points which are non-coplanar with the x axis. Let two planes be drawn, one to include the x axis and **A**, the other, the x axis and **B**. Light synchronization with a clock at the origin will give rise to a relative retardation of $W(x_2' - x_1')/c^2$ between clocks at **B** and **A**. Now generate a new platform by drawing a plane parallel to the x axis and including A and B. Choose an arbitrary origin on this platform and draw a new x axis through it parallel to the original. Then the situation is the same as that discussed previously, hence the each-way speed of light between **A** and **B** will be measured as c.

EXERCISES

1.1 A particle moves relative to the platform of Fig 1.1 with velocity $\overline{V}$ as measured in the aether frame. Its velocity as measured by synchronized clocks is $\overline{u}'$ upon the platform and $\overline{u}$ in the aether frame.

Utilise (1.1-9) with $q \to 0$ to show that

$$u_x = (u_x' + W)/(1 + W u_x'/c^2) \qquad u_x' = (u_x - W)/(1 - W u_x/c^2)$$

Derive the relationship $\sin\theta = \sin\theta'/(1 - W^2\cos^2\theta'/c^2)^{1/2}$ and so show that

$$u_y = u_y'(1 - W^2/c^2)^{1/2}/(1 + W u_x'/c^2) \qquad u_y' = u_y(1 - W^2/c^2)^{1/2}/(1 - W u_x/c^2)$$

Extend the results to three dimensions by resolving u_y and u_y' along normal axes.

1.2 Use the results of the previous exercise to derive

$$u'^2 = c^2 - \frac{(c^2 - u^2)(1 - W^2/c^2)}{(1 - W u_x/c^2)}$$

and so show that if $u = c$ then $u' = c$; $u < c$ then $u' < c$ (for $W < c$)

1.2 The Rate of a Moving Clock

It has been shown in Sec. 1.1 that a clock P which moves rectilinearly with constant self-measured speed from **a** to **b** on the platform shown in Fig. 1.1 will become retarded relative to a clock Q fixed at **a** by

$$\Delta = \frac{R'}{q}\left\{\left(1+\frac{q^2}{c^2}\right)^{1/2} - 1\right\} + \frac{R'W\cos\theta'}{c^2} \qquad (1.2\text{-}1)\ (1.1\text{-}7)$$

where the various symbols have been previously defined.

The two-clock speed of P, say u', as determined by Q together with a fixed clock at **b** light-synchronized with Q, will, in virtue of the associated epoch slip, be given by

$$u' = R'/\left(T' - WR'\cos\theta'/c^2\right) \qquad (1.2\text{-}2)$$

where T' is the elapsed time registered by Q.

But from (1.1-6)

$$\tau' = \frac{R'}{q} = \frac{T'}{\left\{\left(1+\frac{q^2}{c^2}\right)^{1/2} + \frac{qw}{c^2}\cos\theta'\right\}} \qquad \text{since } T' = \tau\left(1 - W^2/c^2\right)^{1/2}$$

hence

$$q\left(T' - WR'\cos\theta'/c^2\right) = R'\left(1+q^2/c^2\right)^{1/2}$$

so that

$$u' = q/\left(1+q^2/c^2\right)^{1/2} \qquad (1.2\text{-}3)$$

and

$$q = u'/\left(1-u'^2/c^2\right)^{1/2} \qquad (1.2\text{-}4)$$

On substituting (2.1-4) in (2.1-1) we get

$$\Delta = \frac{R'}{u'}\left\{1-\left(1-u'^2/c^2\right)^{1/2}\right\} + \frac{R'W\cos\theta'}{c^2}$$

or

$$\Delta = t'\left\{1-\left(1-u'^2/c^2\right)^{1/2}\right\} + Wx'/c^2 \qquad (1.2\text{-}5)$$

where t' is the two-clock time difference between the end points of the movement.

Suppose, now, that P traces out a continuous path[3] which may be treated as the sum of a series of infinitesimal rectilinear steps. Then the differential form of (1.2-5) will apply to each step and the total retardation of P relative to Q will be given by the associated integral.

---

[3] The path need not be planar

Now
$$d\Delta = \left\{1-\left(1-u'^2/c^2\right)^{1/2}\right\}dt' + Wdx'/c^2 \tag{1.2-6}$$

and
$$dt' = dT' - Wdx'/c^2 \tag{1.2-7}$$

hence
$$d\Delta = dT' - \left(1-u'^2/c^2\right)^{1/2} dt' \tag{1.2-8}$$

Then if P traces out a closed curve on leaving Q, its mean rate relative to Q will be given by

$$\int (dT' - d\Delta)\Big/\int dT' = \int \left(1-u'^2/c^2\right)^{1/2} dt'\Big/\int dT'$$

But from (1.2 - 7)
$$\int dt' = \int dT' \qquad \text{since } \oint dx' = 0 \tag{1.2-9}$$

Further, $dt' = ds'/u'$ where ds' is the absolute value of an element of displacement, hence

$$\int (dT' - d\Delta)\Big/\int dT' = \int \frac{\left(1-u'^2/c^2\right)^{1/2}}{u'} ds' \Big/ \int \frac{1}{u'} ds' \tag{1.2-10}$$

This is the general expression for the round-trip rate of a moving clock relative to one fixed on the platform. It is seen to depend only upon speed as measured by stationary synchronized clocks, and is independent of platform velocity relative to the aether. For the case in which u' is piecewise constant (1.2-10) reduces to

$$\int (dT' - d\Delta)\Big/\int dT' = \left(1-u'^2/c^2\right)^{1/2} \tag{1.2-11}$$

It follows that if two clocks P and R, initially set to zero , move with constant speeds u_1' and u_2' in any paths from and back to a stationary clock Q in the same elapsed time $\int dT'$ on Q, then clock P will be retarded relative to R by

$$\left\{\left(1-u_2'^2/c^2\right)^{1/2} - \left(1-u_1'^2/c^2\right)^{1/2}\right\}\int dT' \tag{1.2-12}$$

More generally, if clocks P and R, initially set to zero, move away from each other in any fashion and subsequently come together, then it follows from the integral form of (1.2-8) that P will be retarded relative to R by

$$\int \left(1-u_2'^2/c^2\right)^{1/2} ds_2'/u_2' - \int \left(1-u_1'^2/c^2\right)^{1/2} ds_1'/u_1' \tag{1.2-13}$$

These expressions involve individual speeds as measured on the inertial platform; even in the case of collinear motion it is not possible to express the result in terms of relative speed unless u_1' or $u_2' = 0$.

The instantaneous rate of P relative to Q is given from (1.2-8) by

$$(dT' - d\Delta)/dT' = \left(1-u'^2/c^2\right)^{1/2} dt'/dT'$$

whence, from (1.2-7),

$$\frac{(dT' - d\Delta)}{dT'} = \frac{\left(1 - u'^2/c^2\right)^{1/2}}{\left(1 + Wu'_x/c^2\right)} \tag{1.2-14}$$

This cannot remain constant during the course of the closed movement since u'_x must change sign. We see, further, that the absolute speed W of the platform is involved. At first sight it might be supposed that W could be determined if the reading of P could be transmitted to an observer located at Q (or, what amounts to the same thing, if the observer could see P) and so permit of substitution in (1.2-14). However, this presupposes instantaneous transmission.

Consider the simple case in which clocks P and Q are zeroed together at the origin of platform co-ordinates, and P then proceeds to move at constant speed u' along the positive x axis. When it reaches the point $x' = x'_0$ it reverses its motion and ultimately arrives back at Q. Suppose that on the outward journey a stationary synchronized clock at $x' = x'_1$ reads t'_1 when P reaches this point, and on the return journey a stationary synchronized clock at $x' = x'_2$ reads t'_2 when P reaches it. Further, let the times of arrival of the P readings at the origin be T'_1 and T'_2 respectively, as read on Q. Then at $x' = x'_1$, P reads $\left(1 - u'^2/c^2\right)^{1/2} t'_1$. Also $x'_1 = c\left(T'_1 - t'_1\right)$, whence $T'_1 = t'_1\left(1 + u'/c\right)$.

Hence, for outward movement,

$$\frac{\text{reading of P as observed at Q}}{\text{reading of Q}} = \frac{\left(1 - u'^2/c^2\right)^{1/2}}{\left(1 + u'/c\right)} \tag{1.2-15}$$

The same expression holds for the ratio of incremental readings.

During the return movement P reads $\left(1 - u'^2/c^2\right)^{1/2} t'_2$ at $x' = x'_2$ (the integral form of (1.2-8) holding throughout the movement).

Also $c\left(T'_2 - t'_2\right) = x'_2$ or $T'_2 = t'_2 + x'_2/c$

Further, $t'_2 = x'_0/u' + \left(x'_0 - x'_2\right)/u'$ since $\int dt' = \dfrac{1}{u'}\int ds'$

Hence, for inward movement,

$$\frac{\text{reading of P as observed at Q}}{\text{reading of Q}} = \frac{\left(1 - u'^2/c^2\right)^{1/2}}{\left\{1 + \dfrac{u'}{c}\dfrac{x'_2}{\left(2x'_0 - x'_2\right)}\right\}} \tag{1.2-16}$$

The incremental form is $\dfrac{\left(1 - u'^2/c^2\right)^{1/2} dt'_2}{\left(dt'_2 + dx'_2/c\right)} = \dfrac{\left(1 - u'^2/c^2\right)^{1/2}}{\left(1 - u'/c\right)}$ (1.2-17)

In a similar manner we may show that

for outward movement

$$\frac{\text{reading of Q as observed at P}}{\text{reading of P}} = \frac{(1-u'/c)}{\left(1-u'^2/c^2\right)^{1/2}} \tag{1.2-18}$$

for inward movement

$$\frac{\text{reading of Q as observed at P}}{\text{reading of P}} = \frac{\left\{1-\dfrac{u'}{c}\dfrac{x_2'}{(2x_0'-x_2')}\right\}}{\left(1-u'^2/c^2\right)^{1/2}} \tag{1.2-19}$$

The incremental form is $\dfrac{(1+u'/c)}{\left(1-u'^2/c^2\right)^{1/2}}$ (1.2-20)

It is clear that propagation delay has defeated efforts to deduce W from a comparison of clock readings.

On the basis of the above relationships we may draw up the following table for $u'/c = 0.8$ and $x_0' = 5u'$.

distance of P from origin	adjacent clock reading	reading of P	reading of Q on arrival at P	time of arrival at Q of P's reading
0	0	0	0	0
u'	1	0.6	0.2	1.8
$2u'$	2	1.2	0.4	3.6
$3u'$	3	1.8	0.6	5.4
$4u'$	4	2.4	0.8	7.2
$5u'$	5	3.0	1.0	9.0
$4u'$	6	3.6	2.8	9.2
$3u'$	7	4.2	4.6	9.4
$2u'$	8	4.8	6.4	9.6
u'	9	5.4	8.2	9.8
0	10	6	10	10

A little consideration will show that (1.2-15), (1.2-17) and (1.2-18),(1.2-20) must be closely connected with the doppler formulae for the cases: moving source, fixed receiver and fixed source, moving receiver.

The relevant relationships are developed in greater generality in Sec. 1.4.

1.3 Development of the Lorentz Transformations from the Fitzgerald-Larmor Contractions

It has been shown in Sec 1.1 that if rods and clocks are subject to the Fitzgerald-Larmor contractions when tied to a platform which moves with velocity $\overline{W}$ relative to the aether frame, then, for platform movement in the positive x direction, a clock which is light-synchronized with a master clock located at the origin of platform co-ordinates will be retarded in epoch relative to the master by Wx'/c^2, where x', y', z' are the platform co-ordinates of the synchronized clock.

If the master clock is set to zero when it coincides with the origin of co-ordinates of the aether frame (and other platform clocks are altered by the same amount), and if all aetherial clocks read zero at this time, then, when the aetherial clocks read t, the master clock will read $t\left(1-W^2/c^2\right)^{1/2}$, and the other platform clocks will read

$$t' = t\left(1-W^2/c^2\right)^{1/2} - Wx'/c^2 \qquad (1.3\text{-}1)$$

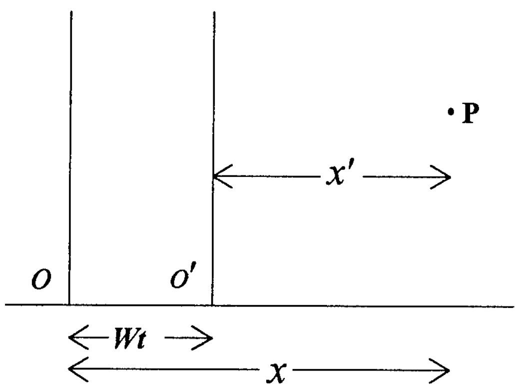

Fig. 1.4

We see that if x and x' are the x co-ordinates of the point **P** (Fig 1.4) when measured in the aether frame and upon the platform at time t by the aetherial clocks, then because of the Fitzgerald contraction of the platform ruler,

$$x' = (x - Wt)\big/\left(1-W^2/c^2\right)^{1/2} \qquad (1.3\text{-}2)$$

On substituting (1.3-2) in (1.3-1) we obtain the clock reading t' on the platform at **P** in terms of the aetherial clock reading at **P**, viz.

$$t' = \left(t - Wx/c^2\right)\big/\left(1-W^2/c^2\right)^{1/2} \qquad (1.3\text{-}3)$$

Since there is no rod compression normal to the direction of motion, $y' = y$ and $z' = z$.

If, now, we replace W by v and write $\left(1-v^2/c^2\right)^{-1/2}$ as β, we obtain the standard form of the Lorentz transformations[5]

[5] The term 'platform' will be replaced by 'frame' from now on, and v will be employed for interframe speed.

$$x' = \beta(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \beta\left(t - vx/c^2\right)$$

The Fitzgerald contraction is often misrepresented in relativist texts by being identified with the reduction in length of an x-orientated rod when measured in a frame in which it is in motion, as compared with its measured length in a frame in which it is stationary. Now if the rod were stationary in the aether frame S, the Fitzgerald contraction, per se, would increase its measured length in another frame S' because of ruler contraction in S'. However, the requirement that end-position readings of the rod be taken at the 'same' time as determined by local synchronized clocks is sufficient to override this effect and give rise to a length reduction. The same requirement is responsible for the reciprocal nature of x-orientated length measurements in S and S'.

In the expression for t' both the Fitzgerald and Larmor contractions are involved, in addition to the epoch slip associated with clock synchronization, Because of these complications we find a reciprocal relationship for time dilatation between frames which would not accompany the Larmor contraction in isolation.

Suppose now that measurements of a common event are carried out in three inertial frames of reference, viz. in S_1 and S_2 which move with constant velocities $\bar{i}v_1$ and $\bar{i}v_2$ with respect to the aether frame S, and within the aether frame itself. Then if the conventional requirements for space and time zeroing are met, the transformations between S_1 and S are given by

(a) $\quad x_1 = \beta_1(x - v_1 t)$ $\quad$ where $\beta_1 = \left(1 - v_1^2/c^2\right)^{-1/2}$

(b) $\quad t_1 = \beta_1\left(t - v_1 x/c^2\right)$

and between S_2 and S by

(c) $\quad x_2 = \beta_2(x - v_2 t)$ $\quad$ where $\beta_2 = \left(1 - v_2^2/c^2\right)^{-1/2}$

(d) $\quad t_2 = \beta_2\left(t - v_2 x/c^2\right)$

From (a) and (b) we obtain the reciprocal relationships

(e) $\quad x = \beta_1(x_1 + v_1 t_1)$

(f) $\quad t = \beta_1\left(t_1 + v_1 x_1/c^2\right)$

On substituting (e) and (f) in (c) we find that

$$x_2 = \beta_1\beta_2\left\{x_1\left(1 - v_1 v_2/c^2\right) - (v_2 - v_1)t_1\right\}$$

or

$$x_2 = \frac{x_1 - (v_2 - v_1)t_1/(1 - v_1v_2/c^2)}{\left\{(1 - v_1^2/c^2)(1 - v_2^2/c^2)/(1 - v_1v_2/c^2)^2\right\}^{1/2}}$$

$$= \frac{x_1 - (v_2 - v_1)t_1/(1 - v_1v_2/c^2)}{\left\{1 - (v_2 - v_1)^2/c^2(1 - v_1v_2/c^2)^2\right\}^{1/2}}$$

i.e. $x_2 = \beta_{12}(x_1 - v_{12}t_1)$ where $\beta_{12} = (1 - v_{12}^2/c^2)^{-1/2}$

and v_{12} is the speed of frame 2 as measured in frame 1.

Similarly $t_2 = \beta_{12}(t_1 - v_{12}\,x_1/c^2)$

Thus the Lorentz transformations hold not only between the aether frame S and other inertial frames S_1 and S_2 in relative motion along a common x axis, but also between S_1 and S_2 themselves. Correspondingly, the transformations are said to exhibit a group property.

1.4 The Doppler Effect

We will first develop the doppler formula within the aether frame, where the speed of propagation is c in all directions and synchronized clocks show the same time at a given instant.

Corresponding to a source velocity $\overline{u}$ and a receiver velocity $\overline{w}$, let the co-ordinates at time t be

source: $X_o + u_x t$ $\quad Y_o + u_y t$ $\quad Z_o + u_z t$ (1.4-1)

receiver: $x_o + w_x t$ $\quad y_o + w_y t$ $\quad z_o + w_z t$ (1.4-2)

A pulse leaving the source at time t_1 is received at time $t_1 + r_1/c$, where

$$r_1^2 = \sum\left\{x_o + w_x(t_1 + r_1/c) - X_o - u_x t_1\right\}^2 \tag{1.4-3}$$

If the next pulse leaves the source at time t_2 it is received at $t_2 + r_2/c$. Then if f_s is the source frequency and f_r is the received frequency

$$f_s = 1/(t_2 - t_1) \quad \text{and} \quad f_r = 1\Big/\left\{t_2 - t_1 + \frac{1}{c}(r_2 - r_1)\right\}$$

$$\text{or} \quad f_r = f_s\Big/\left(1 + \frac{1}{c}\frac{dr}{dt}\right) \tag{1.4-4}$$

But from (1.4-3)

$$2r\frac{dr}{dt} = 2\sum\left\{x_o + w_x\left(t+\frac{r}{c}\right) - X_o - u_x t\right\}\left(w_x - u_x + \frac{1}{c}w_x\frac{dr}{dt}\right)$$

or $$\frac{dr}{dt} = \sum l\left(w_x - u_x + \frac{1}{c}w_x\frac{dr}{dt}\right)$$

where l is the x direction cosine of the vector $\bar{r}$ directed from the source at the time of emission to the receiver at the time of reception, hence

$$\frac{dr}{dt}\left(1 - w_r/c\right) = w_r - u_r \tag{1.4-5}$$

and $$f_r = f_s \Big/ \left\{1 + \frac{1}{c}\frac{\left(w_r - u_r\right)}{\left(1 - w_r/c\right)}\right\} = f_s\frac{\left(1 - w_r/c\right)}{\left(1 - u_r/c\right)} \tag{1.4-6}$$

where w_r and u_r are the resolved parts of $\bar{w}$ and $\bar{u}$ along $\bar{r}$.

This is the standard doppler formula for medium propagation.

Suppose now that observations are carried out in an inertial frame S' which moves with velocity $\bar{i}v$ relative to the aether frame.

Since $t' = \beta\left(t - vx/c^2\right)$ we have, at the receiver, $\Delta t' = \beta\Delta t\left(1 - vw_x/c^2\right)$, hence

$$f_r' = f_r \big/ \beta\left(1 - vw_x/c^2\right) \tag{1.4-7}$$

Similarly $$f_s' = f_s \big/ \beta\left(1 - vu_x/c^2\right) \tag{1.4-8}$$

so that $$\frac{f_r'}{f_s'} = \frac{f_r}{f_s}\frac{\left(1 - vu_x/c^2\right)}{\left(1 - vw_x/c^2\right)} = \frac{\left(1 - w_r/c\right)\left(1 - vu_x/c^2\right)}{\left(1 - u_r/c\right)\left(1 - vw_x/c^2\right)} \tag{1.4-9}$$

(1.4-9) may be expressed in terms of measured values in S' by means of transformations developed elsewhere[6], involving the resolved part of velocity along a line joining a retarded source position to a point of reception.

The appropriate transformations are

$$\begin{aligned}\left(1 - u_{r'}'/c\right) &= \left(1 - u_r/c\right) \big/ \beta^2\left(1 - vu_x/c^2\right)\left(1 - l\,v/c\right)\\ \left(1 - w_{r'}'/c\right) &= \left(1 - w_r/c\right) \big/ \beta^2\left(1 - vw_x/c^2\right)\left(1 - l\,v/c\right)\end{aligned} \tag{1.4-10}$$

where l is the x direction cosine of the above-mentioned line.

Substitution of (1.4-10) in (1.4-9) yields

6 F.A.P.T Pt. 2. Sec. 1.3

$$f'_r = f'_s \frac{(1 - w'_{r'}/c)}{(1 - u'_{r'}/c)} \tag{1.4-11}$$

This is simply a primed form of (1.4-6).
The result could have been anticipated since the measured speed of light is the same in all directions in S' and of equal value to that in the aether frame, so that the same analysis applies. It must, of course, be borne in mind that the values assigned to f'_r and f'_s are based on the readings of stationary synchronized clocks in S' to which the receiver and source are adjacent at any time.

The frequency of the source when in motion in the aether frame is related to its frequency f_o when stationary by

$$f_s = f_o\left(1 - u^2/c^2\right)^{1/2} \tag{1.4-12}$$

Then from (1.4-8) $\quad f'_s = f_o\left(1 - u^2/c^2\right)^{1/2} \Big/ \beta\left(1 - v\,u_x/c^2\right)$

Now it may be shown that[6]

$$\left(1 - u'^2/c^2\right)^{1/2} = \left(1 - u^2/c^2\right)^{1/2} \Big/ \beta\left(1 - v\,u_x/c^2\right) \tag{1.4-13}$$

hence $\quad f'_s = f_o\left(1 - u'^2/c^2\right)^{1/2} \quad (1.4\text{-}14)$

(1.4-14) is the primed form of (1.4-12), the 'rest' value of source frequency having the same measured value f_o in all frames of reference.

(1.4-6) and (1.4-11) may be expressed in terms of f_o as

$$f_r = f_o\left(1 - u^2/c^2\right)^{1/2} \frac{(1 - w_r/c)}{(1 - u_r/c)} \tag{1.4-15}$$

$$f'_r = f_o\left(1 - u'^2/c^2\right)^{1/2} \frac{(1 - w'_{r'}/c)}{(1 - u'_{r'}/c)} \tag{1.4-16}$$

For the particular case of a receiver stationary in S'[7]

$$f'_r = f_o \frac{\left(1 - u'^2/c^2\right)^{1/2}}{(1 - u'_{r'}/c)} \tag{1.4-17}$$

---

7 (1.4-17) is presented in texts on special relativity as *the* doppler formula. From the nature of its derivation in such texts, the components of the expression are necessarily evaluated in a frame of reference in which the receiver is stationary and in which, in consequence, the absolute velocity of the source is equal to its velocity relative to the receiver. This appears to have given rise to the mistaken impression that only relative velocity need be involved in the doppler formula.

If the line joining the retarded position of the source to a stationary receiver is normal to $\bar{u}$, i.e. if the source is seen to be along a line normal to its path, $u'_{r'} = 0$ and

$$f'_r = f_o\left(1 - u'^2/c^2\right)^{1/2} \tag{1.4-18}$$

This formula is said to express the transverse doppler effect, but this is clearly a misnomer. It was first experimentally verified by Ives and Stilwell in 1938.

So far, all timing has been based on the readings of local, stationary, synchronized clocks. We may, however, express the result in the case of a moving receiver in somewhat different form by computing the received frequency by means of a clock carried with the receiver. The associated time intervals are then reduced by the factor $\left(1 - w'^2/c^2\right)^{1/2}$[8] so that the received frequency is given from (1.4-16) by

$$f'_r = f_o \frac{\left(1 - u'^2/c^2\right)^{1/2}\left(1 - w'_{r'}/c\right)}{\left(1 - w'^2/c^2\right)^{1/2}\left(1 - u'_{r'}/c\right)} \tag{1.4-19}$$

This introduces a greater symmetry into the expression but at the expense of employing a stationary clock to compute f_o, stationary synchronized clocks to compute u' and w' and a moving clock to compute f'_r. In these circumstances it is easily shown that f'_r/f_o has the same value when the source approaches a stationary receiver or the receiver approaches a stationary source, at a given speed. But it follows from (1.4-16) and (1.4-19) that, in general, the frequency ratio cannot be expressed in terms of the relative velocity of source and receiver.

EXERCISE

1.3 A turntable of radius R_1 rotates with angular velocity ω, as measured by a stationary clock at the centre. A source is located on the table at a distance R_2 from the centre and a receiver is located at random on the periphery. If the source frequency is measured as f_o by a co-moving clock, show that the received frequency is given to at least a second order in v/c by

(a) $$f_r = f_o \frac{\left(1 - R_2^2\,\omega^2/c^2\right)^{1/2}}{\left(1 - R_1^2\,\omega^2/c^2\right)^{1/2}}$$ when measured by a clock which moves with the receiver.

(b) $$f_r = f_o\left(1 - R_2^2\,\omega^2/c^2\right)^{1/2}$$ when measured by stationary clocks adjacent to the periphery.

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8 This follows from equation (1.2-8).

CHAPTER 2

INTERFRAME TRANSFORMATIONS IN ELECTRODYNAMICS

In this chapter we examine the manner in which the values of various electrical and mechanical point functions change when measurements of a given source complex are transferred from the aether frame S to a frame S' which moves with velocity $\bar{i}v$ relative to S.

In virtue of the group property exhibited by these functions (Appendix 2), the same transformation formulae must hold between all inertial frames in relative motion along a common x axis.

It is shown by direct transformation of the individual components that the Maxwell-Lorentz equations for a moving medium maintain the same (Lorentz-covariant) form in all inertial frames of reference.

2.1 Transformation of the Density Functions ρ, $\overline{\mathbf{J}}$, $\overline{\mathbf{P}}$ and $\overline{\mathbf{M}}$

(1) Transformation of charge and current density

It follows directly from the population density transformations[1] for a set of sources translating with velocity $\overline{w}$ in S, that in S'

$$\rho' = \rho\beta\left(1 - vw_x/c^2\right) = \beta\left(\rho - vJ_x/c^2\right)$$

$$J'_x = \rho' w'_x = \rho\beta\left(1 - vw_x/c^2\right)\left(w_x - v\right)/\left(1 - vw_x/c^2\right) = \rho\beta\left(w_x - v\right) = \beta\left(J_x - \rho v\right)$$

$$J'_y = \rho' w'_y = \rho\beta\left(1 - vw_x/c^2\right)w_y/\beta\left(1 - vw_x/c^2\right) = \rho w_y = J_y$$

Similarly $J'_z = J_z$

1 See F.A.P.T. Pt.2. Sec.1.7. Although the Lorentz transformations were employed in Pt.2. in a mapping capacity to generate a secondary source system from a given primary source, the associated transformations apply equally to the case of a single source system when viewed from different reference frames.

In summary

$$J'_x = \beta\left(J_x - \rho v\right) \qquad J'_y = J_y \qquad J'_z = J_z \qquad \rho' = \beta\left(\rho - vJ_x/c^2\right) \tag{2.1-1}$$

whence

$$J_x = \beta\left(J'_x + \rho' v\right) \qquad J_y = J'_y \qquad J_z = J'_z \qquad \rho = \beta\left(\rho' + vJ'_x/c^2\right) \tag{2.1-2}$$

(2) Transformation of doublet polarisation density $\overline{\mathbf{P}}$

Suppose that the doublet complex translates with velocity $\overline{u}$ in S. Then the dimensions of an individual doublet will transform as follows[1]:

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\beta\left(1 - vu_x/c^2\right)} \quad ; \quad y'_2 - y'_1 = y_2 - y_1 + \frac{vu_y}{c^2}\frac{\left(x_2 - x_1\right)}{\left(1 - vu_x/c^2\right)} \quad ;$$
$$z'_2 - z'_1 = z_2 - z_1 + \frac{vu_z}{c^2}\frac{\left(x_2 - x_1\right)}{\left(1 - vu_x/c^2\right)}$$

whence

$$p'_x = \frac{p_x}{\beta\left(1 - vu_x/c^2\right)} \quad ; \quad p'_y = p_y + \frac{vu_y}{c^2}\frac{p_x}{\left(1 - vu_x/c^2\right)} \quad ;$$
$$p'_z = p_z + \frac{vu_z}{c^2}\frac{p_x}{\left(1 - vu_x/c^2\right)} \tag{2.1-3}$$

But $\Delta\tau' = \Delta\tau/\beta\left(1 - vu_x/c^2\right)$ where $\Delta\tau$ is an element of volume which translates with velocity $\overline{u}$ in S, hence

$$P'_x = P_x \;;\; P'_y = \beta\left(1 - vu_x/c^2\right)P_y + \beta v u_y P_x/c^2 \;;$$
$$P'_z = \beta\left(1 - vu_x/c^2\right)P_z + \beta v u_z P_x/c^2 \tag{2.1-4}$$

and

$$P_x = P'_x \;;\; P_y = \beta\left(1 + vu'_x/c^2\right)P'_y - \beta v u'_y P'_x/c^2 \;;$$
$$P_z = \beta\left(1 + vu'_x/c^2\right)P'_z - \beta v u'_z P'_x/c^2 \tag{2.1-5}$$

where $u'_x = \left(u_x - v\right)/\left(1 - vu_x/c^2\right)$ = x component of velocity in S'.

(3) Transformation of whirl moment density $\overline{\mathbf{M}}$

Suppose that the whirl complex translates with velocity $\overline{u}$ in S.

Let a circulating charge arrive at a point **P** on an individual whirl at a time t in S, the corresponding co-ordinates being $X+u_x t$, $Y+u_y t$, $Z+u_z t$. In S' the time at **P** is read as $\beta\left(t-v\left(X+u_x t\right)/c^2\right)$. If the following charge arrives at **P** at time $t+\Delta t$ in S, the time, as read in S', is $\beta\left\{t+\Delta t-v\left(X+u_x(t+\Delta t)\right)/c^2\right\}$ so that in S' the interval between successive arrivals is given by $\Delta t'=\Delta t\beta\left(1-vu_x/c^2\right)$, whence the circulating current relationship is

$$I'=I/\beta\left(1-vu_x/c^2\right) \tag{2.1-6}$$

The whirl is supposed neutral in S[2].

Charges which are fixed within the whirl when viewed in S will remain fixed when viewed in S', so that the total negative (stationary) charge between any two points of the whirl will appear equal in S and S'.

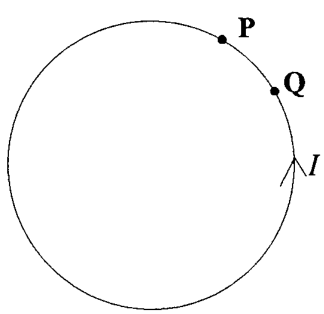

Fig. 2.1

Suppose that a positive charge q_1 arrives at **P** and a positive charge q_2 arrives at **Q** at the time t in S. (Fig 2.1) Then as measured in S', q_1 arrives at **P** at the time $\beta\left(t-v\left(X+u_x t\right)/c^2\right)$, say t', while q_2 arrives at **Q** at the time $\beta\left(t-v\left(X+u_x t+\Delta x\right)/c^2\right)$, i.e. $\beta v\Delta x/c^2$ earlier than q_1 arrives at **P**. (Δx is the x component of PQ as measured in S).

Then by the time t', an additional charge $\beta v\Delta x\, I'/c^2$ will have passed **Q** in the direction of **P**, so that the total net charge between **P** and **Q** when measured at the common time t' in S' will be $\beta v\Delta x\, I'/c^2$.

Now $\Delta x'=\Delta x/\beta\left(1-vu_x/c^2\right)$ and $I'=I/\beta\left(1-vu_x/c^2\right)$ hence $\Delta xI'=\Delta x'I$ and $\lambda'=\beta v\Delta x'\, I/c^2\Delta s'$ where λ' is the net line density of charge between **P** and **Q** as measured in S' at the arbitrary time t', and $\Delta s'$ is the length **PQ** as measured in S'.

Then

$$\lambda'=\frac{-\beta vI}{c^2}\frac{\overline{i}\cdot\Delta\overline{s}'}{\Delta s'} \tag{2.1-7}$$

where $\Delta\overline{s}'$ has the direction of current flow.

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2 If the whirl should exhibit polarisation in S resulting from a variable net line density of charge with zero total charge, it is easily shown that the transformation of this component of polarisation will take the same form as that developed for the discrete doublet distribution, and should therefore be included with it in (2.1-4). This should not be confused with whirl polarisation which is observed in S' but not in S.

The total net charge , as measured in S', is easily shown to be zero, hence the polarisation associated with the whirl in S' is given by

$$\bar{p}' = \oint \bar{s}'\lambda' ds' = \frac{-\beta v I}{c^2} \oint \bar{s}'\bar{i} \cdot d\bar{s}' \qquad (2.1\text{-}8)$$

where $\bar{s}'$ is the position vector of an element of the contour relative to an arbitrary origin.

It may be shown[3] that

$$\oint \bar{s}'\bar{i} \cdot d\bar{s}' = \bar{i} \times \bar{S}' \qquad (2.1\text{-}9)$$

where $\bar{S}'$ is the vector area of the whirl as measured in S'.

Then

$$\bar{p}' = \frac{-\beta v I}{c^2} \left(\bar{i} \times \bar{S}'\right) \qquad (2.1\text{-}10)$$

It is shown in Appendix 1 that

$$\bar{S}' = \bar{i}\left\{ S_x - \frac{v/c^2}{\left(1 - v u_x/c^2\right)}\left(u_y S_y + u_z S_z\right)\right\} + \bar{j}\frac{S_y}{\beta\left(1 - v u_x/c^2\right)} + \bar{k}\frac{S_z}{\beta\left(1 - v u_x/c^2\right)} \qquad (2.1\text{-}11)$$

whence, from (2.1-10),

$$p'_x = 0 \qquad p'_y = \frac{v}{c}\frac{m_z}{\left(1 - v u_x/c^2\right)} \qquad p'_z = -\frac{v}{c}\frac{m_y}{\left(1 - v u_x/c^2\right)} \qquad (2.1\text{-}12)$$

and

$$P'_x = 0 \qquad P'_y = \beta\frac{v}{c}M_z \qquad P'_z = -\beta\frac{v}{c}M_y \qquad (2.1\text{-}13)$$

Since $\bar{m}' = I'\bar{S}'/c$, we have from (2.1-6) and (2.1-11)

$$\left.\begin{aligned} m'_x &= \frac{I}{c\beta\left(1 - v u_x/c^2\right)}\left\{ S_x - \frac{v/c^2}{\left(1 - v u_x/c^2\right)}\left(u_y S_y + u_z S_z\right)\right\} \\ m'_y &= \frac{I S_y}{c\beta^2\left(1 - v u_x/c^2\right)^2} \qquad m'_z = \frac{I S_z}{c\beta^2\left(1 - v u_x/c^2\right)^2} \end{aligned}\right\} \qquad (2.1\text{-}14)$$

[3] F.A.P.T. Pt. 2. Sec. 1.5b

whence

$$\left.\begin{aligned} M'_x &= M_x - \frac{v/c^2}{\left(1-vu_x/c^2\right)}\left(u_y M_y + u_z M_z\right) \\ M'_y &= \frac{M_y}{\beta\left(1-vu_x/c^2\right)} \qquad M'_z = \frac{M_z}{\beta\left(1-vu_x/c^2\right)} \end{aligned}\right\} \quad (2.1\text{-}15)$$

and

$$\left.\begin{aligned} M_x &= M'_x + \frac{v/c^2}{\left(1+vu'_x/c^2\right)}\left(u'_y M'_y + u'_z M'_z\right) \\ M_y &= \frac{M'_y}{\beta\left(1+vu'_x/c^2\right)} \qquad M_z = \frac{M'_z}{\beta\left(1+vu'_x/c^2\right)} \end{aligned}\right\} \quad (2.1\text{-}16)$$

(4) Overall polarisation transformation

The total polarisation as measured in S' derives in part from discrete doublet polarisation in S and possible whirl polarisation in S; it is given by (2.1-4). In addition, there is the component of whirl polarisation which appears in S' but not in S, viz. (2.1-13). By addition, we have

$$\left.\begin{aligned} P'_x &= P_x \\ P'_y &= \beta\left(1-vu_x/c^2\right)P_y + \beta v u_y P_x/c^2 + \beta v M_z/c \\ P'_z &= \beta\left(1-vu_x/c^2\right)P_z + \beta v u_z P_x/c^2 - \beta v M_y/c \end{aligned}\right\} \quad (2.1\text{-}17)$$

On combining (2.1-17) with (2.1-15) we obtain the inverse relationships

$$\left.\begin{aligned} P_x &= P'_x \\ P_y &= \beta\left(1+vu'_x/c^2\right)P'_y - \beta v u'_y P'_x/c^2 - \beta v M'_z/c \\ P_z &= \beta\left(1+vu'_x/c^2\right)P'_z - \beta v u'_z P'_x/c^2 + \beta v M'_y/c \end{aligned}\right\} \quad (2.1\text{-}18)$$

EXERCISE

2.1 Extend the analysis of Sec. 2.1 to cover the case in which $\overline{P}$ and $\overline{M}$ are time-dependent. Show that if whirl current has the same value at all points of the contour when measured at a given time in S, it will vary around the periphery when measured at a given time in S', but that the same transformation formulae will hold provided that the whirl moment $\overline{m}'$ is defined as $\displaystyle Lim_{s'\to 0}\frac{1}{2c}\oint \overline{s}' \times I' d\overline{s}'$, where $\overline{s}'$ is the position vector of an element of the contour relative to a point about which the contour shrinks.

2.2 Transformation of ϕ, $\overline{\mathbf{A}}$, $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$[4]

Let **O** $(x_0\ y_0\ z_0)$ be a point of evaluation of the retarded potentials ϕ and $\overline{A}$ in the aether frame S at time t_0, and let **Q** $(x_1\ y_1\ z_1)$ be the appropriately-retarded position of a source element at time t_1.

Then if $\mathbf{OQ} = R$

$$R = c(t_0 - t_1)$$

If the corresponding space and time co-ordinates of **O** and **Q** when measured in S' are $x_0'\ y_0'\ z_0'\ t_0'$ and $x_1'\ y_1'\ z_1'\ t_1'$, and $\mathbf{OQ} = R'$, then

$$R' = c(t_0' - t_1')$$

since the measured speed of propagation in S' is c.

Further

$$\begin{aligned} x_0' - x_1' &= \beta\left(x_0 - x_1 - v(t_0 - t_1)\right) \\ y_0' - y_1' &= y_0 - y_1 \\ z_0' - z_1' &= z_0 - z_1 \\ t_0' - t_1' &= \beta\left(t_0 - t_1 - v(x_0 - x_1)/c^2\right) \end{aligned}$$

If l, m, n and l', m', n' are the direction cosines of $\overrightarrow{\mathbf{QO}}$ when measured in S and S'

$$R' = c(t_0' - t_1') = \beta c\left(t_0 - t_1 - v(x_0 - x_1)/c^2\right) = \beta R(1 - l\,v/c) \qquad (2.2\text{-}1)$$

Also

$$l'R' = (x_0' - x_1') = \beta\left(x_0 - x_1 - v(t_0 - t_1)\right) = \beta R(l - v/c)$$

hence

$$l' = (l - v/c)/(1 - l\,v/c) \qquad (2.2\text{-}2)$$

Similarly, we find that

$$m' = m/\beta(1 - lv/c) \qquad n' = n/\beta(1 - lv/c) \qquad (2.2\text{-}3)$$

Let the source strength at **Q** be q. Then if $\overline{u}$ and $\overline{u}'$ represent source velocity, as measured in S and S', the corresponding microscopic retarded potentials at **O** will be given by

4 The analysis in this section is, in large part, formally identical with that developed in F.A.P.T. Pt.2, Sec. 1.3, but the physical context is different. Here we are concerned with measurements carried out in two frames of reference, S and S', upon a common set of source elements; in Pt. 2 we were concerned with measurements conducted at conjugate points upon two different source systems, designated S and S', within a common frame of reference.

$$\phi = \frac{q}{R\left(1-u_R/c\right)} \qquad \phi' = \frac{q}{R'\left(1-u'_{R'}/c\right)} \tag{2.2-4}$$

$$\overline{A} = \frac{q\overline{u}}{c\,R\left(1-u_R/c\right)} \qquad \overline{A}' = \frac{q\overline{u}'}{c\,R'\left(1-u'_{R'}/c\right)} \tag{2.2-5}$$

Now $$u_R = lu_x + mu_y + nu_z \quad \text{and} \quad u'_{R'} = l'u'_x + m'u'_y + n'u'_z \tag{2.2-6}$$

whence we find that

$$\left(1-u'_{R'}/c\right) = \left(1-u_R/c\right)\Big/\beta^2\left(1-lv/c\right)\left(1-vu_x/c^2\right) \tag{2.2-7}$$

Then $$\phi' = \beta\left(\phi - vA_x/c\right) \tag{2.2-8}$$

$$A'_x = \beta\left(A_x - v\phi/c\right) \qquad A'_y = A_y \qquad A'_z = A_z \tag{2.2-9}$$

and $$\phi = \beta\left(\phi' + v\,A'_x/c\right) \tag{2.2-10}$$

$$A_x = \beta\left(A'_x + v\phi'/c\right) \qquad A_y = A'_y \qquad A_z = A'_z \tag{2.2-11}$$

These microscopic relationships, which have been developed for a single source, must continue to hold, by superposition, for any combination of sources, since the relationships are linear, and consequently apply to singlet, doublet and whirl configurations moving in any manner.

Maxwell's equations for material media are written in terms of macroscopic quantities[5]. In particular, the macroscopic $\overline{E}$ and $\overline{B}$ fields continue to be defined by

$$\overline{E} = -grad\,\phi - \frac{1}{c}\frac{\partial \overline{A}}{\partial t} \qquad \text{and} \quad \overline{B} = curl\,\overline{A} \tag{2.2-11}$$

but ϕ and $\overline{A}$ are now macroscopic potentials.

For present purposes the macroscopic potential at a point within a source complex may be defined to be the microscopic potential at that point of all source elements outside a small spheroidal surface centred upon that point, the dimensions of the spheroid being sufficiently great to ensure that the potential at the centre varies smoothly when the surface is translated. It is clear that the greater the statistical regularity of the fine structure of the complex, the smaller the excluding surface may be made. It is conventionally supposed that, while satisfying the above condition, the surface may be so reduced in size that further reduction does not significantly alter the potential at the centre. It follows that the relationships between ϕ and ϕ', and $\overline{A}$ and $\overline{A}'$ are the same as those for the microscopic potentials, since the same sources are involved in each frame of reference.

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5 The relationships between macroscopic and microscopic quantities are discussed in detail in F.A.P.T. Pt.1. Secs. 5.17/18

The relationships between the macroscopic forms of $\overline{E}$ and $\overline{E}'$ and of $\overline{B}$ and $\overline{B}'$ are consequently identical with their microscopic counterparts. The relevant analysis has been carried out elsewhere[6], although in a different context (footnote 4), with the following results:

$$\begin{array}{lll} E'_x = E_x & E'_y = \beta\left(E_y - \frac{v}{c}B_z\right) & E'_z = \beta\left(E_z + \frac{v}{c}B_y\right) \\ B'_x = B_x & B'_y = \beta\left(B_y + \frac{v}{c}E_z\right) & B'_z = \beta\left(B_z - \frac{v}{c}E_y\right) \end{array} \qquad (2.2\text{-}12)$$

$$\begin{array}{lll} E_x = E'_x & E_y = \beta\left(E'_y + \frac{v}{c}B'_z\right) & E_z = \beta\left(E'_z - \frac{v}{c}B'_y\right) . \\ B_x = B'_x & B_y = \beta\left(B'_y - \frac{v}{c}E'_z\right) & B_z = \beta\left(B'_z + \frac{v}{c}E'_y\right) \end{array} \qquad (2.2\text{-}13)$$

2.3 Transformation of the Maxwell-Lorentz Equations for Moving Media

Maxwell's equations for a medium (doublet/whirl complex) at rest in the aether frame[7] have been extended in an earlier analysis[8] to cover the case of uniform motion. The modified equations take the Lorentzian form

$$div\ \overline{E} = 4\pi\left(\rho - div\ \overline{P}\right) \qquad (2.3\text{-}1)$$

$$curl\ \overline{E} = -\frac{1}{c}\frac{\partial \overline{B}}{\partial t} \qquad (2.3\text{-}2)$$

$$div\ \overline{B} = 0 \qquad (2.3\text{-}3)$$

$$curl\overline{B} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial \overline{E}}{\partial t} + \frac{4\pi}{c}\frac{\partial \overline{P}}{\partial t} + 4\pi\, curl\overline{M} + \frac{4\pi}{c}curl\left(\overline{P}\times\overline{u}\right) \qquad (2.3\text{-}4)$$

where $\overline{u}$ is the velocity of the complex in the aether frame[9].

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6 F.A.P.T. Pt. 2. Sec. 1.4.

7 For a detailed development see F.A.P.T. Pt. 1, Ch. 5.

8 F.A.P.T. Pt. 2. The treatment employs the Lorentz transformations, but in a mapping capacity only, and is consequently independent of the 'relativistic' considerations of previous sections.

9 For analytical reasons the x axis of co-ordinates was chosen to coincide with the direction of doublet/whirl movement in the development of the above equations. However, the latter are seen to be invariant with respect to choice of axes and will therefore hold for an arbitrary direction of $\overline{u}$ within a set of axes so chosen that S' moves with velocity $\overline{i}v$ relative to S.

The values to be assigned to $\overline{P}$ and $\overline{M}$ are those obtaining during the movement[10].

We are now in a position to carry out the transformations of (2.3-1) to (2.3-4) between the aether frame S and the inertial frame S', by making use of the transformations developed in Sec. 2.1 for $\rho, \overline{J}, \overline{P}$ and $\overline{M}$ and in Sec. 2.2 for $\overline{E}$ and $\overline{B}$. These are repeated below for convenience.

The algebraic working involved in the transformations is extremely tedious and intermediate steps have, for the most part, been omitted.

$$J_x = \beta\left(J'_x + \rho' v\right) \qquad J_y = J'_y \qquad J_z = J'_z \qquad \rho = \beta\left(\rho' + v J'_x/c^2\right) \tag{2.3-5}$$

$$\left.\begin{aligned} P_x &= P'_x \\ P_y &= \beta\left(1 + v u'_x/c^2\right)P'_y - \beta v u'_y P'_x/c^2 - \beta v M'_z/c \\ P_z &= \beta\left(1 + v u'_x/c^2\right)P'_z - \beta v u'_z P'_x/c^2 + \beta v M'_y/c \end{aligned}\right\} \tag{2.3-6}$$

$$\left.\begin{aligned} &M_x = M'_x + \frac{v/c^2}{\left(1 + v u'_x/c^2\right)}\left(u'_y M'_y + u'_z M'_z\right) \\ &M_y = \frac{M'_y}{\beta\left(1 + v u'_x/c^2\right)} \qquad M_z = \frac{M'_z}{\beta\left(1 + v u'_x/c^2\right)} \end{aligned}\right\} \tag{2.3-7}$$

$$E_x = E'_x \qquad E_y = \beta\left(E'_y + \frac{v}{c}B'_z\right) \qquad E_z = \beta\left(E'_z - \frac{v}{c}B'_y\right) \tag{2.3-8}$$

$$B_x = B'_x \qquad B_y = \beta\left(B'_y - \frac{v}{c}E'_z\right) \qquad B_z = \beta\left(B'_z + \frac{v}{c}E'_y\right) \tag{2.3-9}$$

The appropriate transformations for the derivatives are

$$\frac{\partial}{\partial x} = \beta\left\{\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right\} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \qquad \frac{\partial}{\partial t} = \beta\left\{\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right\} \tag{2.3-10}$$

On substituting (2.3-8), (2.3-9) and (2.3-10) in the x component of (2.3-2) we find that

$$\left(curl'\overline{E}'\right)_x - \frac{v}{c}div'\overline{B}' = -\frac{1}{c}\frac{\partial B'_x}{\partial t'} \tag{2.3-11}$$

10 The variation of $\overline{P}$ and $\overline{M}$ which derives from movement is discussed in CH.3. Such variation is irrelevant to present considerations.

and from (2.3-3)

$$div'\overline{B}' - \frac{v}{c^2}\frac{\partial B'_x}{\partial t'} - \frac{v}{c}\left(curl'\overline{E}'\right)_x = 0 \tag{2.3-12}$$

On combining (2.3-11) and (2.3-12) we get

$$\left(curl'\overline{E}'\right)_x = -\frac{1}{c}\frac{\partial B'_x}{\partial t'} \tag{2.3-13}$$

and

$$div'\overline{B'} = 0 \tag{2.3-14}$$

Substitution of (2.3-8), (2.3-9) and (2.3-10) in the y and z components of (2.3-2) yields

$$\left(curl'\overline{E}'\right)_y = -\frac{1}{c}\frac{\partial B'_y}{\partial t'} \qquad \text{and} \qquad \left(curl'\overline{E}'\right)_z = -\frac{1}{c}\frac{\partial B'_z}{\partial t'} \tag{2.3-15}$$

On transforming (2.3-1) in accordance with (2.3-5), (2.3-6), (2.3-8) and (2.3-10) we find that

$$\left.\begin{aligned}
&div'\overline{E}' + \frac{v}{c}\left(curl'\overline{B}'\right)_x \\
&= \frac{v}{c}\left\{\frac{4\pi}{c}J'_x + \frac{1}{c}\frac{\partial E'_x}{\partial t'} + \frac{4\pi}{c}\frac{\partial P'_x}{\partial t'} + 4\pi\left(curl'\overline{M}'\right)_x + \frac{4\pi}{c}\left(curl'\left(\overline{P}'\times\overline{u}'\right)\right)_x\right\} \\
&+4\pi\left(\rho' - div'\overline{P}'\right)
\end{aligned}\right\} \tag{2.3-16}$$

and on transforming the x component of (2.3-4) via (2.3-5/9) we obtain

$$\left.\begin{aligned}
\left(curl'\overline{B}'\right)_x &= \frac{4\pi}{c}J'_x + \frac{1}{c}\frac{\partial E'_x}{\partial t'} + \frac{4\pi}{c}\frac{\partial P'_x}{\partial t'} + 4\pi\left(curl'\overline{M}'\right)_x + \frac{4\pi}{c}\left(curl'\left(\overline{P}'\times\overline{u}'\right)\right)_x \\
&- \frac{v}{c}\left\{div'\overline{E}' - 4\pi\left(\rho' - div'\overline{P}'\right)\right\}
\end{aligned}\right\} \tag{2.3-17}$$

Then on combining (2.3-16) and (2.3-17) we find that

$$\left\{div'\overline{E}' - 4\pi\left(\rho' - div'\overline{P}'\right)\right\}/\beta^2 = 0$$

hence

$$div'\overline{E}' = 4\pi\left(\rho' - div'\overline{P}'\right) \tag{2.3-18}$$

and

$$\left(curl'\overline{B}'\right)_x = \frac{4\pi}{c}J'_x + \frac{1}{c}\frac{\partial E'_x}{\partial t'} + \frac{4\pi}{c}\frac{\partial P'_x}{\partial t'} + 4\pi\left(curl'\overline{M}'\right)_x + \frac{4\pi}{c}\left(curl'\left(\overline{P}'\times\overline{u}'\right)\right)_x \tag{2.3-19}$$

In like manner, transformation of the y and z components of (2.3-4) give rise to primed versions of identical form.

Then

$$curl'\overline{B}' = \frac{4\pi}{c}\overline{J}' + \frac{1}{c}\frac{\partial \overline{E}'}{\partial t'} + \frac{4\pi}{c}\frac{\partial \overline{P}'}{\partial t'} + 4\pi\, curl'\,\overline{M}' + \frac{4\pi}{c} curl'\left(\overline{P}' \times \overline{u}'\right) \quad (2.3\text{-}20)$$

It follows from (2.3-13/14/15/18/20) that the Maxwell-Lorentz equations for a medium in motion take the same form in all inertial frames under a Lorentz transformation, i.e. they are Lorentz-covariant.

Before proceeding to Sec. 2.4 it is of interest to review the procedures which have brought us to the present position.

Maxwell's equations for a medium at rest were developed in "Field Analysis and Potential Theory, Part 1" as a consequence of an 'electronic' model. This comprised statistically-regular configurations of singlets, doublets and whirls, the doublet and whirl centres being at rest in a frame of reference in which the speed of propagation, or retardation constant, was c in all directions (aether). The density functions $\rho, \overline{J}, \overline{P}$ and $\overline{M}$ and the macroscopic retarded potentials ϕ and $\overline{A}$ were defined in terms of the geometry and kinematics of the model, while $\overline{E}$ and $\overline{B}$ were expressed in terms of ϕ and $\overline{A}$.

In Part 2 the Lorentz transformations were employed in a mapping capacity to extend Maxwell's equations to the case of a medium in uniform translation in the aether frame.

In Part 3 use has been made of the Lorentz transformations to investigate the manner in which the measured values of the density functions and $\overline{E}$ and $\overline{B}$ change when viewed in a non-aetherial inertial frame, and to demonstrate the Lorentz-covariance of the Maxwell-Lorentz equations for a moving medium.

Nowhere in the above developments have experimental results of an electrical nature been incorporated, and to this extent it can be claimed that Maxwell's equations, as presented here, relate to mathematics rather than physics. In particular, no force law has been assumed for charge interaction; correspondingly, it has not been required that the source systems under consideration should comprise physically-viable configurations. As has been pointed out in the preface to Part 1, the physical aspects are introduced by way of the Lorentz force formula and through experimentally-determined parameters in the constitutive equations.

Electrokinetic considerations appear, for the first time, in the following section.

2.4 Transformation of Electrical and Mechanical Force

We suppose that within the aether frame S, the momentum of a particle of velocity $\bar{u}$ takes the form

$$\bar{p} = \frac{m\bar{u}}{\left(1-u^2/c^2\right)^{1/2}} \tag{2.4-1}$$

where m is the mass of the particle.

This is the third fundamental assumption that has been made. The other two - the Fitzgerald-Larmor length and time modifications resulting from motion through the aether - have provided the bases of the foregoing 'relativistic' treatment of kinematics; the momentum formula permits of the corresponding development of kinetics.

Then on the assumption that the Lorentz force formula holds in S, we now proceed to show that it will hold in another inertial frame S' only if momentum in that frame is ascribed the value

$$\bar{p}' = \frac{m\bar{u}'}{\left(1-u'^2/c^2\right)^{1/2}} \tag{2.4-2}$$

where $\bar{u}'$ is the velocity in S'[11].

Consider first the y component of the force which acts upon a charge q of mass m moving with velocity $\bar{u}$ in the aether frame. We can equate the y component of the Lorentz force with the corresponding rate of change of mechanical momentum,

i.e.

$$q\left\{E_y + \frac{1}{c}\left(\bar{u}\times\bar{B}\right)_y\right\} = \frac{d}{dt}\frac{mu_y}{\left(1-u^2/c^2\right)^{1/2}} \tag{2.4-3}$$

The left hand side of (2.4-3) may be expressed in terms of the values of $\bar{E}'$ and $\bar{B}'$ which would be measured at q in the frame S' which moves with velocity $\bar{i}v$ in S.

This becomes

$$q\left\{\beta\left(E'_y + \frac{v}{c}B'_z\right) + \frac{u'_z B'_x/c}{\beta\left(1+vu'_x/c^2\right)} - \frac{\left(u'_x+v\right)/c}{\left(1+vu'_x/c^2\right)}\beta\left(B'_z + \frac{v}{c}E'_y\right)\right\}$$

which reduces to

$$\frac{q}{\beta\left(1+vu'_x/c^2\right)}\left\{E'_y + \frac{1}{c}\left(\bar{u}'\times\bar{B}'\right)_y\right\} \tag{2.4-4}$$

---

11 The transformation formula for momentum is commonly derived by the evaluation, in two inertial frames, of the linear momenta of lossless colliding particles, together with the assumption that momentum is conserved in each frame. Given this, the following analysis demonstrates that if the Lorentz force formula holds in one inertial frame, it holds in all such frames.

Since

$$dt' = \beta\, dt\left(1 - vu_x/c^2\right)$$
and
$$\left(1-u^2/c^2\right)^{1/2} = \left(1-u'^2/c^2\right)^{1/2} \Big/ \beta\left(1+vu'_x/c^2\right) \tag{2.4-5}$$

the right hand side of (2.4-3) may be expressed as

$$\beta\left(1-vu_x/c^2\right)\frac{d}{dt'}\frac{mu'_y}{\left(1-u'^2/c^2\right)^{1/2}} = \frac{1}{\beta\left(1+vu'_x/c^2\right)}\frac{d}{dt'}\frac{mu'_y}{\left(1-u'^2/c^2\right)^{1/2}} \tag{2.4-6}$$

whence from (2.4-4) and (2.4-6)

$$q\left\{E'_y + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)_y\right\} = \frac{d}{dt'}\frac{mu'_y}{\left(1-u'^2/c^2\right)^{1/2}} \tag{2.4-7}$$

A similar analysis obtains for the z component.

For the x component in S

$$q\left\{E_x + \frac{1}{c}\left(\overline{u}\times\overline{B}\right)_x\right\} = \frac{d}{dt}\frac{mu_x}{\left(1-u^2/c^2\right)^{1/2}} \tag{2.4-8}$$

The left hand side may be expressed as

$$q\left\{E'_x + \frac{u'_y\left(B'_z + \frac{v}{c}E'_y\right)}{c\left(1+vu'_x/c^2\right)} - \frac{u'_z\left(B'_y - \frac{v}{c}E'_z\right)}{c\left(1+vu'_x/c^2\right)}\right\}$$

which reduces to

$$\frac{q}{\left(1+vu'_x/c^2\right)}\left\{E'_x + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)_x + \frac{v}{c^2}\overline{u}'\cdot\overline{E}'\right\}$$

$$= \frac{q}{\left(1+vu'_x/c^2\right)}\left\{E'_x + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)_x + \frac{v}{c^2}\overline{u}'\cdot\left(\overline{E}' + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)\right)\right\}$$

$$= q\Bigg[E'_x + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)_x + \frac{v/c^2}{\left(1+vu'_x/c^2\right)}\left\{u'_y\left(E'_y + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)_y\right) + u'_z\left(E'_z + \frac{1}{c}\left(\overline{u}'\times\overline{B}'\right)_z\right)\right\}\Bigg] \tag{2.4-9}$$

On applying (2.4-5) to the right hand side of (2.4-8) we get

$$\frac{1}{\left(1+vu_x'/c^2\right)}\left\{\frac{d}{dt'}\frac{mu_x'}{\left(1-u'^2/c^2\right)^{1/2}}+\frac{d}{dt'}\frac{mv}{\left(1-u'^2/c^2\right)^{1/2}}\right\} \tag{2.4-10}$$

The second term within the brackets is equal to $\dfrac{mv}{\left(1-u'^2/c^2\right)^{3/2}}\dfrac{u'}{c^2}\dfrac{du'}{dt'}$

It is easily shown by expansion that

$$\sum u_x'\frac{d}{dt'}\frac{mu_x'}{\left(1-u'^2/c^2\right)^{1/2}}=mu'\frac{du'}{dt'}\frac{1}{\left(1-u'^2/c^2\right)^{3/2}}$$

hence (2.4-10) may be written as

$$\frac{1}{\left(1+vu_x'/c^2\right)}\left\{\frac{d}{dt'}\frac{mu_x'}{\left(1-u'^2/c^2\right)^{1/2}}+\frac{v}{c^2}\sum u_x'\frac{d}{dt'}\frac{mu_x'}{\left(1-u'^2/c^2\right)^{1/2}}\right\}$$

$$=\frac{d}{dt'}\frac{mu_x'}{\left(1-u'^2/c^2\right)^{1/2}}+\frac{v/c^2}{\left(1+vu_x'/c^2\right)}\left\{u_y'\frac{d}{dt'}\frac{mu_y'}{\left(1-u'^2/c^2\right)^{1/2}}+u_z'\frac{d}{dt'}\frac{mu_z'}{\left(1-u'^2/c^2\right)^{1/2}}\right\} \tag{2.4-11}$$

Then on substituting (2.4-7) and its z equivalent in (2.4-11) and equating with (2.4-9) we obtain

$$q\left\{E_x'+\frac{1}{c}\left(\bar{u}'\times\bar{B}'\right)_x\right\}=\frac{d}{dt'}\frac{mu_x'}{\left(1-u'^2/c^2\right)^{1/2}} \tag{2.4-12}$$

It follows from (2.4-7) and (2.4-12) that momentum in S' must be ascribed the value $m\bar{u}'/\left(1-u'^2/c^2\right)^{1/2}$ if the Lorentz force formula is to be applied in that frame. Mass has the same value in all inertial frames of reference.

We see from (2.4-4), (2.4-6), (2.4-9) and (2.4-11) that both electrical and mechanical forces as measured in S and S' are related by

$$F_x=F_x'+\frac{v/c^2}{\left(1+vu_x'/c^2\right)}\left(u_y'F_y'+u_z'F_z'\right) \tag{2.4-13}$$

$$F_y=\frac{F_y'}{\beta\left(1+vu_x'/c^2\right)} \qquad F_z=\frac{F_z'}{\beta\left(1+vu_x'/c^2\right)} \tag{2.4-14}$$

As is usual, the inverse relationships are obtained by interchange of primed and unprimed quantities and the reversal of the sign of v.

It will be observed that the transformation formulae for the components of $\overline{M}$ (2.1-16) are formally identical with those for the components of electrical and mechanical force. It is shown in Appendix 1 that this applies also to the components of area. Then the establishment of the group property for one establishes it for the others.

EXERCISES

2.2 By differentiating the x component of (2.4-1) with respect to time, show that

$$\frac{du_x}{dt} = \left\{ F_x - \frac{mu_x}{\left(1-u^2/c^2\right)^{3/2}} \frac{u}{c^2} \frac{du}{dt} \right\} \frac{\left(1-u^2/c^2\right)^{1/2}}{m}$$

whence

$$\text{acceleration} = \left\{ \overline{F} - \frac{\overline{u}}{c^2}(\overline{u} \cdot \overline{F}) \right\} \frac{(1-u^2/c^2)^{1/2}}{m}$$

It will be observed that no simple meaning can be attached to the concept of 'inertial mass' (where acceleration = force/inertial mass) since acceleration is dependent upon the direction of application of the force.
[The terms 'rest mass' (m_0) and 'relativistic inertial mass' $\left(m_0/\left(1-u^2/c^2\right)^{1/2}\right)$ have now dropped out of favour.]

2.3 If the momentum transformation formula is derived independently of electrical considerations, and $\overline{E}$ and $\overline{B}$ are defined in any frame of reference by means of the Lorentz force formula (rather than through the agency of the retarded potentials), make use of the electrical forms of (2.4-13) and (2.4-14) to derive the transformations for $\overline{E}$ and $\overline{B}$ between different frames.

[Set various component velocities to zero to develop component transformations, and reinsert these to confirm that the transformations hold in general.]

CHAPTER 3

VELOCITY DEPENDENCE WITHIN A SINGLE FRAME OF REFERENCE

3.1 General Considerations

In the previous chapter we were concerned, *inter alia*, with the manner in which the density functions $\rho, \bar{J}, \bar{P}$ and $\bar{M}$ change value when measured in different frames of reference. We were nowhere concerned with possible changes which would be observed in a given frame as a result of bulk movement of the source complex in that frame. The two cases are quite distinct. In the former we have a given source complex and two viewing platforms; in the latter, one platform and possibly two source configurations. We now address the second case.

Consider the most general form of source system (comprising individual charges) in the aether frame, and let it be subject to electrical forces only, as expressed by the Lorentz formula. From this source system we generate a second system within the aether frame by an application of the Lorentz transformation; thus a charge q in the primary system having the co-ordinates x, y, z, t gives rise to a charge q' of equal magnitude having the co-ordinates $\beta(x-vt), y, z, \beta\left(t-vx/c^2\right)$[1].

Then when the path of q is given, the path of q' is defined. In this context v is simply an arbitrary constant[2].

Let $\mathbf{O}$ $\left(x_0\, y_0\, z_0\, t_0\right)$ and $\mathbf{O}'\left(x_0'\, y_0'\, z_0'\, t_0'\right)$ be positions and times of evaluation of the potentials in the two systems, where the space and time co-ordinates are related by the Lorentz transformation. If $\mathbf{Q}$ is the appropriately-retarded position of q for evaluation of its contribution to the potentials at $\mathbf{O}$ at time t_0 and its co-ordinates are $\left(x_1\, y_1\, z_1\, t_1\right)$ then $t_1 = t_0 - R/c$ where $R = \mathbf{OQ}$. Since $\mathbf{Q}$ lies on the path of q, the corresponding point $\mathbf{Q}'$

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1 Note that one system of axes and one set of synchronized clocks are employed for both the primary and secondary source systems.

2 It will be observed that we have now applied the Lorentz transformation in three different ways:

(1) as in the purely mathematical mapping procedure employed in F.A.P.T. Pt. 2 to establish the Maxwell-Lorentz equations for a moving medium.
(2) as in the present instance, where the mapping procedure generates a new physical system.
(3) as in CH. 2 where the transformation relates measurements made on a single source system in two frames of reference.

lies on the path of q' and has the co-ordinates $\beta\left(x_1 - vt_1\right)\, y_1\, z_1\, \beta\left(t_1 - vx_1/c^2\right)$. Given this, it has been shown in "Field Analysis and Potential Theory, Part 2" that $\mathbf{Q}'$ is the appropriately-retarded position of q' for evaluation of its contribution to the potentials at $\mathbf{O}'$ at time t_0'. It has also been shown that, as a consequence, the associated field relationships at $\mathbf{O}$ and $\mathbf{O}'$ are given by

$$E_x' = E_x \qquad E_y' = \beta\left(E_y - \frac{v}{c}B_z\right) \qquad E_z' = \beta\left(E_z + \frac{v}{c}B_y\right) \tag{3.1-1}$$

$$B_x' = B_x \qquad B_y' = \beta\left(B_y + \frac{v}{c}E_z\right) \qquad B_z' = \beta\left(B_z - \frac{v}{c}E_y\right) \tag{3.1-2}$$

These relationships apply to all paired sources in turn and, by superposition, to complete primary and secondary source systems.

Suppose now that $\mathbf{O}$ and $\mathbf{O}'$ are identified with the locations of corresponding charges q_0 and $q_0'\left(= q_0\right)$. Since q_0 is subject to Lorentz forces only, we have

$$q_0\left\{\overline{E} + \frac{1}{c}\left(\overline{u} \times \overline{B}\right)\right\} = \frac{d}{dt}\frac{m\overline{u}}{\left(1 - u^2/c^2\right)^{1/2}} \tag{3.1-3}$$

where m is the mass of q_0 and $\overline{u}$ is its velocity.

Then on applying (3.1-1), (3.1-2) and the mathematical treatment of Sec. 2.4 to the present situation we find that (3.1-3) leads to

$$q_0'\left\{\overline{E}' + \frac{1}{c}\left(\overline{u}' \times \overline{B}'\right)\right\} = \frac{d}{dt'}\frac{m\overline{u}'}{\left(1 - u'^2/c^2\right)^{1/2}} \tag{3.1-4}$$

We see that the Lorentz force which acts upon q_0' at time t_0' is just that which is required to maintain its rate of change of momentum, when its motion is defined by that of its parent charge q_0 at time t_0 via the agency of the Lorentz transformation.

Since the same argument holds for all paired charges, it follows that the secondary system represents a possible self-contained space-time configuration subject only to Lorentz forces, and that we may generate as many secondary systems as we please from a given primary system by choice of v.

It is clear that the viability of the secondary configuration will not be affected if shifted as a whole in space and time, so that we may equally well transform the primary system according to

$$x, y, z, t \rightarrow x^\Diamond, y^\Diamond, z^\Diamond, t^\Diamond \tag{3.1-5}$$

where

$$x^\Diamond = \beta\left(x - vt\right) + X \qquad y^\Diamond = y + Y \qquad z^\Diamond = z + Z \qquad t^\Diamond = \beta\left(t - vx/c^2\right) + T$$

and X, Y, Z, T are constants having the same values for all source elements.

We now restrict considerations to source configurations which exhibit an identifiable bulk velocity, e.g. a moving medium comprising a doublet/whirl complex in which the doublet and whirl centres translate together. If conduction currents are present they should be confined to paths which share the doublet/whirl velocity. It will be further supposed that the associated density functions are invariant or periodic in time.

Let the bulk velocity of the primary system be $\bar{u}$ and that of the secondary system $\bar{u}^{\Diamond}$,

so that

$$u_x^{\Diamond} = (u_x - v)/(1 - vu_x/c^2) \qquad u_y^{\Diamond} = u_y/\beta(1 - vu_x/c^2) \qquad u_z^{\Diamond} = u_z/\beta(1 - vu_x/c^2) \quad (3.1\text{-}6)$$

Then provided that the configuration of the secondary system is a unique function of its velocity as a whole, we are led to conclude that if the primary system is forced to alter its velocity in the aether frame from $\bar{u}$ to $\bar{u}^{\Diamond}$, it will rearrange its space/time[3] geometry in accordance with (3.1-5). Under these conditions the x component of spacing between doublet and whirl centres is reduced $\beta(1 - vu_x/c^2)$ times, while the period of time-dependent processes is increased by the same factor. In particular, for a system initially at rest in the aether frame and subsequently in motion with speed v, there will be a compression of β times in the direction of motion and an equal increase in cyclic period.

It will be observed that in arriving at these conclusions the Fitzgerald-Larmor contractions have not been invoked[4]. Whereas such contractions have been pivotal in the development of the Lorentz transformations, as applied to interframe measurement, they have played no part in the present analysis, where the Lorentz transformations have been arbitrarily chosen to relate the primary and secondary source configurations. Thus the conclusions reached lend credence to the Fitzgerald-Larmor proposals, insofar as material systems may be considered to derive from charge complexes subject to Lorentz forces.

Suppose now that the secondary configuration in the aether frame S is viewed in a frame S' which moves with velocity $-\bar{i}v$ relative to S. Then

$$x' = \beta(x^{\Diamond} + vt^{\Diamond}) \qquad y' = y^{\Diamond} \qquad z' = z^{\Diamond} \qquad t' = \beta(t^{\Diamond} + vx^{\Diamond}/c^2) \quad (3.1\text{-}7)$$

which, on substitution from (3.1-5), yields[5]

$$x' = x + \beta(X + vT) \quad y' = y + Y \quad z' = z + Z \quad t' = t + \beta(T + vX/c^2) \quad (3.1\text{-}8)$$

---

3 Here we employ the term 'space/time' in lieu of 'spatial configuration as a function of time'.

4 On the other hand, the basic expression for momentum within the aether frame has been an essential part of the analysis.

5 If the origins of co-ordinates in S and S' do not coincide at zero time in each frame, additional constants will appear in (3.1-7) and (3.1-8).

Hence for this particular value of velocity relative to S, the space/time sequences of individual sources, as measured by synchronized clocks in S', will be identical with those measured in S but with different space and time origins - a result to be expected since we have, in effect, applied a forward and reverse transformation to the source complex. Likewise, the bulk velocities in the two frames will be equal.

It follows that if the bulk velocity of a given, self-contained, time-invariant or periodic source system has the same measured value on different occasions in any two inertial frames of reference, the differential space/time behaviour of the two configurations will be measured as identical. Correspondingly, the associated electromagnetic analyses must agree in all respects; this follows automatically when the retardation constant is identified with the measured speed of light in each frame, viz. c.

EXERCISES

3.1 A horizontal platform moves with velocity $\bar{i}v$ in the aether frame. A star is observed from a point on the x axis of the platform to lie in the xz plane and to have an angle of elevation ϕ (as measured from the positive x direction), while the corresponding angle, as measured at a stationary, coincident point in the aether frame, is ϕ_o.

Make use of Fig. 1.3 and equations (1.1-10a) and (1.1-3), with θ' replaced by ϕ, to derive the aberration relationships

$$\cos\phi_0 = \frac{\cos\phi - v/c}{1 - v\cos\phi/c} \qquad \cos\phi = \frac{\cos\phi_0 + v/c}{1 + v\cos\phi_0/c}$$

Show that these expressions continue to apply when the aetherial frame is replaced by any inertial frame, and $\bar{i}v$ is the platform velocity relative to it.

3.2 Derive the above relationships by utilising the appropriate direction - cosine transformation developed in Sec.2.2, and employ the considerations of Sec.3.1 to show that the measured value of aberration angle is unchanged when the phase velocity of light within the viewing telescope is altered, e.g. by filling it with water (Airy's experiment).

3.2 Modification of the Constitutive Equations

The constitutive equations provide a link between the density functions $\overline{J}, \overline{P}$ and $\overline{M}$, and the potential derivatives $\overline{E}$ and $\overline{B}$, and so permit of a nexus between a purely mathematical development and a physical theory[7].

In a Lorentzian context the relationships of primary interest for a medium at rest are those which connect $\overline{J}$ and $\overline{E}$, $\overline{P}$ and $\overline{E}$, and $\overline{M}$ and $\overline{B}$[8].

The auxiliary point functions $\overline{D}$ and $\overline{H}$ are defined by

$$\overline{D} = \overline{E} + 4\pi\,\overline{P} \qquad \overline{H} = \overline{B} - 4\pi\,\overline{M} \tag{3.2-1}$$

For an isotropic medium at rest, we write

$$\overline{D} = K\overline{E} \qquad \overline{B} = \mu\overline{H} \tag{3.2-2}$$

where K is known as the permittivity (dielectric constant) and μ the permeability[9].

Then

$$\overline{P} = \frac{K-1}{4\pi}\overline{E} = k_e\overline{E} \qquad \overline{M} = \frac{\mu-1}{4\pi\mu}\overline{B} = k_m\overline{B} \tag{3.2-3}$$

where k_e and μk_m are the electric and magnetic susceptibilities.

We have also

$$\overline{J} = \sigma\overline{E} \tag{3.2-4}$$

where σ represents conductivity.

In view of the conclusions reached in the previous section we need consider only those modifications of the constitutive equations which accompany movement in the aether frame, since the same electrical parameters are measured at a given point of a particular source complex in all inertial frames when exhibiting the same bulk velocity (or when at rest).

Let the source system, initially at rest (as indicated by the superscript 0), begin to move with uniform velocity $\overline{u}$ in the aether frame. Suppose that ancillary axes X Y Z are set up such that $\overline{u}$ is directed along the positive X axis. Then the source parameters will be modified by the motion in the manner prescribed by the Lorentz transformation, where v is replaced by $-u$.

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7 See p. 29 above, and Preface to F.A.P.T. Pt.1.

8 That such linkage exists must be demonstrated experimentally.

9 In anisotropic media $\overline{D}$ is not collinear with $\overline{E}$ nor $\overline{B}$ with $\overline{H}$, so that K and μ cannot be represented by simple constants. These complications do not concern us here.

Thus, from (2.1-17)

$$P_X = P_X^0 \qquad P_Y = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(P_Y^0 - \frac{u}{c}M_Z^0\right) \quad P_Z = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(P_Z^0 + \frac{u}{c}M_Y^0\right) \quad (3.2\text{-}5)$$

From (2.1-15)

$$M_X = M_X^0 \qquad M_Y = \left(1-u^2/c^2\right)^{1/2} M_Y^0 \qquad M_Z = \left(1-u^2/c^2\right)^{1/2} M_Z^0 \quad (3.2\text{-}6)$$

From (2.1-2)

$$J_X^0 = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(J_X - \rho u\right) \qquad J_Y^0 = J_Y \qquad J_Z^0 = J_Z$$
$$\rho^0 = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(\rho - uJ_X/c^2\right) \quad (3.2\text{-}7)$$

From (2.2-13)

$$E_X^0 = E_X \qquad E_Y^0 = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(E_Y - \frac{u}{c}B_Z\right) \quad E_Z^0 = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(E_Z + \frac{u}{c}B_Y\right) \quad (3.2\text{-}8)$$

and

$$B_X^0 = B_X \qquad B_Y^0 = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(B_Y + \frac{u}{c}E_Z\right) \quad B_Z^0 = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\left(B_Z - \frac{u}{c}E_Y\right) \quad (3.2\text{-}9)$$

Then by substitution of the components of $\overline{P}^0 = k_e\overline{E}^0$ and $\overline{M}^0 = k_m\overline{B}^0$ in (3.2-5) and substitution of (3.2-8) and (3.2-9) in the result, we obtain, to a first order in u/c,

$$P_X = k_eE_X \qquad P_Y \approx k_eE_Y - \left(k_e + k_m\right)\frac{u}{c}B_Z \qquad P_Z \approx k_eE_Z + \left(k_e + k_m\right)\frac{u}{c}B_Y \quad (3.2\text{-}10)$$

Substitution of the components of $\overline{M}^0 = k_m\overline{B}^0$ in (3.2-6) and substitution of (3.2-9) in the result yields

$$M_X = k_mB_X \qquad M_Y = k_m\left(B_Y + \frac{u}{c}E_Z\right) \quad M_Z = k_m\left(B_Z - \frac{u}{c}E_Y\right) \quad (3.2\text{-}11)$$

Then

$$\overline{P} \approx k_e\overline{E} + \left(k_e + k_m\right)\frac{1}{c}\left(\overline{u}\times\overline{B}\right) \approx \frac{(K-1)}{4\pi}\overline{E} + \frac{(K\mu - 1)}{4\pi\mu}\frac{1}{c}\left(\overline{u}\times\overline{B}\right) \quad (3.2\text{-}12)$$

$$\overline{M} = k_m\left(\overline{B} - \frac{1}{c}\left(\overline{u}\times\overline{E}\right)\right) = \frac{(\mu - 1)}{4\pi\mu}\left(\overline{B} - \frac{1}{c}\left(\overline{u}\times\overline{E}\right)\right) \quad (3.2\text{-}13)$$

On writing $\bar{J}^0 = \sigma \bar{E}^0$ in (3.2-7) and substituting from (3.2-8) we get to a first order in u/c

$$J_X \approx \sigma E_X + \rho u \quad J_Y \approx \sigma\left(E_Y - \frac{u}{c}B_Z\right) \quad J_Z \approx \sigma\left(E_Z + \frac{u}{c}B_Y\right) \tag{3.2-14}$$

whence
$$\bar{J} \approx \rho\bar{u} + \sigma\left(\bar{E} + \frac{1}{c}(\bar{u} \times \bar{B})\right) \tag{3.2-15}$$

It will be seen that the term $\rho\bar{u}$ represents the convection current density. The conduction current component, which is clearly of greater significance when dealing with closed circuits, is given by

$$\bar{J}_{conduc.} \approx \sigma\left(\bar{E} + \frac{1}{c}(\bar{u} \times \bar{B})\right) \tag{3.2-16}$$

CHAPTER 4

A CRITIQUE OF CONVENTIONAL RELATIVISTIC ELECTROMAGNETICS

Minkowski's relativistic treatment of electromagnetics, as presented in current textbooks, begins with Maxwell's equations for a material medium at rest in some reference frame Σ, viz.

$$div\,\overline{D} = 4\pi\rho \tag{4.1}$$

$$curl\,\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t} \tag{4.2}$$

$$div\,\overline{B} = 0 \tag{4.3}$$

$$curl\,\overline{H}^{*} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial\overline{D}}{\partial t}^{1} \tag{4.4}$$

The derivative transformations, corresponding to measurement in a frame Σ', moving with velocity $\overline{i}v$ relative to Σ, are

$$\frac{\partial}{\partial x} = \beta\left\{\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right\} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \quad \frac{\partial}{\partial t} = \beta\left\{\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right\} \tag{4.5}$$

On applying (4.5) to the cartesian components of (4.2) and to (4.3) we get

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = -\frac{1}{c}\beta\left\{\frac{\partial B_x}{\partial t'} - v\frac{\partial B_x}{\partial x'}\right\} \tag{4.6}$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial}{\partial x'}\beta\left(E_z + \frac{v}{c}B_y\right) = -\frac{1}{c}\frac{\partial}{\partial t'}\beta\left(B_y + \frac{v}{c}E_z\right) \tag{4.7}$$

$$\frac{\partial}{\partial x'}\beta\left(E_y - \frac{v}{c}B_z\right) - \frac{\partial E_x}{\partial y'} = -\frac{1}{c}\frac{\partial}{\partial t'}\beta\left(B_z - \frac{v}{c}E_y\right) \tag{4.8}$$

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1 $\overline{H}^{*}$ is written with a starred superscript to distinguish it from the Lorentzian $\overline{H}$. The two are unequal in a frame of reference in which the medium is in motion. [As a consequence, the associated constitutive equations assume different forms.]

$$\beta\left\{\frac{\partial B_x}{\partial x'}-\frac{v}{c^2}\frac{\partial B_x}{\partial t'}\right\}+\frac{\partial B_y}{\partial y'}+\frac{\partial B_z}{\partial z'}=0 \tag{4.9}$$

Elimination of $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial t'}$ in turn from (4.6) and (4.9) yields

$$\frac{\partial}{\partial y'}\beta\left(E_z+\frac{v}{c}B_y\right)-\frac{\partial}{\partial z'}\beta\left(E_y-\frac{v}{c}B_z\right)=-\frac{1}{c}\frac{\partial B_x}{\partial t'} \tag{4.10}$$

$$\frac{\partial B_x}{\partial x'}+\frac{\partial}{\partial y'}\beta\left(B_y+\frac{v}{c}E_z\right)+\frac{\partial}{\partial z'}\beta\left(B_z-\frac{v}{c}E_y\right)=0 \tag{4.11}$$

It is seen from (4.7), (4.8), (4.10) and (4.11) that (4.2) and (4.3) assume their primed forms when the following substitutions are made:

$$E'_x=E_x \qquad E'_y=\beta\left(E_y-\frac{v}{c}B_z\right) \qquad E'_z=\beta\left(E_z+\frac{v}{c}B_y\right) \tag{4.12}$$

$$B'_x=B_x \qquad B'_y=\beta\left(B_y+\frac{v}{c}E_z\right) \qquad B'_z=\beta\left(B_z-\frac{v}{c}E_y\right) \tag{4.13}$$

In like manner, application of (4.5) to the components of (4.4) and to (4.1) gives rise to the primed forms of (4.1) and (4.4), provided that

$$D'_x=D_x \qquad D'_y=\beta\left(D_y-\frac{v}{c}H^*_z\right) \qquad D'_z=\beta\left(D_z+\frac{v}{c}H^*_y\right) \tag{4.14}$$

$$H^{*\prime}_x=H^*_x \qquad H^{*\prime}_y=\beta\left(H^*_y+\frac{v}{c}D_z\right) \qquad H^{*\prime}_z=\beta\left(H^*_z-\frac{v}{c}D_y\right) \tag{4.15}$$

$$J'_x=\beta\left(J_x-\rho v\right) \qquad J'_y=J_y \qquad J'_z=J_z \qquad \rho'=\beta\left(\rho-\frac{v}{c^2}J_x\right) \tag{4.16}$$

Textbooks now advance the following argument: "According to the Principle of Relativity, the laws of physics have the same form in all inertial frames of reference. Maxwell's equations are Lorentz-covariant between Σ and Σ', given (4.12) to (4.16), hence they must hold in this form in Σ'."

Passing over the question as to whether Maxwell's equations, in isolation, represent laws of physics[2], we see that the argument hinges upon the Principle of Relativity; without it, Lorentz-covariance would lose its significance.

This approach is quite different to that of Sec. 2.3, where Maxwell's equations for a medium at rest or in motion within a single frame of reference (aether) were known

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[2] Footnote 7, p. 39.

ab initio, and where the interframe transformations were accomplished by utilising transformations previously deduced for both the density functions and $\overline{E}$ and $\overline{B}$ on the basis of the model; the Principle of Relativity was nowhere invoked. Nevertheless, the results are in conformity with it. In particular, the interframe velocity $\overline{v}$ vanishes and the velocity $\overline{u}$ of the material medium (i.e. the doublet/whirl complex) appears in the same mathematical context in each frame, as required.

In the Minkowski scheme $\overline{P}$ and $\overline{M}^*$ are defined by

$$\overline{P} = \left(\overline{D} - \overline{E}\right)/4\pi \qquad \text{and} \qquad \overline{M}^* = \left(\overline{B} - \overline{H}^*\right)/4\pi \tag{4.17}$$

while $\overline{P}'$ and $\overline{M}^{*'}$ are defined by the primed expressions.

Then substitution in (4.12) to (4.15) yields

$$P_x' = P_x \quad P_y' = \beta\left(P_y + \frac{v}{c}M_z^*\right) \quad P_z' = \beta\left(P_z - \frac{v}{c}M_y^*\right) \tag{4.18}$$

$$M_x^{*'} = M_x^* \quad M_y^{*'} = \beta\left(M_y^* - \frac{v}{c}P_z\right) \quad M_z^{*'} = \beta\left(M_z^* + \frac{v}{c}P_y\right) \tag{4.19}$$

Substitution of (4.17) and its primed equivalent in (4.4) and its primed equivalent leads to the Minkowski relationships

$$curl\,\overline{B} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial \overline{E}}{\partial t} + \frac{4\pi}{c}\frac{\partial \overline{P}}{\partial t} + 4\pi\, curl\,\overline{M}^* \qquad \text{in } \Sigma \tag{4.20}$$

and

$$curl'\,\overline{B}' = \frac{4\pi}{c}\overline{J}' + \frac{1}{c}\frac{\partial \overline{E}'}{\partial t'} + \frac{4\pi}{c}\frac{\partial \overline{P}'}{\partial t'} + 4\pi\, curl'\,\overline{M}^{*'} \qquad \text{in } \Sigma' \tag{4.21}$$

while the corresponding Lorentzian forms are

$$curl\,\overline{B} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial \overline{E}}{\partial t} + \frac{4\pi}{c}\frac{\partial \overline{P}}{\partial t} + 4\pi\, curl\,\overline{M} \qquad \text{in } \Sigma \text{ (where } \overline{u} = 0) \tag{4.22}$$

and

$$curl'\,\overline{B}' = \frac{4\pi}{c}\overline{J}' + \frac{1}{c}\frac{\partial \overline{E}'}{\partial t'} + \frac{4\pi}{c}\frac{\partial \overline{P}'}{\partial t'} + 4\pi\, curl'\,\overline{M}' + \frac{4\pi}{c}curl'\left(\overline{P}' \times \left(-\overline{i}v\right)\right) \tag{4.23}$$

Then we may identify $\overline{M}^*$ with $\overline{M}$ in Σ, where the medium is at rest, and $\overline{M}^{*'}$ with $\overline{M}' + \frac{1}{c}\left(\overline{P}' \times \left(-\overline{i}v\right)\right)$ in Σ' where the medium velocity is $-\overline{i}v$, since the same transformations hold for $\overline{E}, \overline{B}, \overline{J}$ and $\overline{P}$ in each scheme. Relativists claim that $\overline{M}^{*'}$ represents the observed value of whirl moment density in Σ' on the grounds that a doublet of moment $\overline{p}$, when translating with velocity $\overline{u}$, gives rise to a whirl moment $\overline{m}$, where $\overline{m} = \frac{1}{c}\left(\overline{p} \times \overline{u}\right)$. This is clearly incorrect, as has been pointed our elsewhere[3]. Not

3 F.A.P.T. Pt. 2, Sec. 1.9.

only are the associated vector potentials incompatible, but, in the case of a normally-translating doublet, the radial $\overline{B}$ field of the whirl along its axis is twice that of the doublet, while the field of the whirl along the line of the doublet is half that of the doublet (Ex. 4.2). Attempts to interpret $\overline{M}^{*'}$ on the basis of an electronic model are doomed to failure; within this context $\overline{M}^{*'}$ remains a hybrid which permits of an illusion of Lorentz-covariance between Σ and Σ' when the particular choice, $\overline{u} = 0$ in Σ, requires the suppression of the medium velocity factor in (4.21).

The Minkowski treatment poses a further problem which is most clearly demonstrated in connection with the constitutive equations. These are written down for a medium at rest in Σ and developed for a medium moving in Σ'. But we are concerned, not with the development of a constitutive formula for a moving medium in terms of its stationary parameter in a different reference frame, but in terms of its stationary parameter in the same frame. While it is always tacitly assumed that the same quantitative relationship holds for a given medium, when stationary in any inertial frame, it is not obvious that the Principle of Relativity can be extended this far. In contrast, it has been shown in Sec. 3.1 that not only do the same constitutive equations apply at a given source element in different frames of reference for a given medium velocity, but also that, in these circumstances, the measured values of the density functions and of $\overline{E}$ and $\overline{\mathrm{B}}$ will be identical.

Relativist literature is replete with asymmetrical analyses - an inevitable concomitant of a choice of reference frame in which some component of the system under consideration is at rest. Such analyses are presumably responsible for the mistaken belief that only relative velocities are significant in physical interactions. This has already been remarked upon in connection with the rates of moving clocks and the doppler formulae; it continues to be pertinent in the field of charge/charge and whirl/charge interaction.

EXERCISES

4.1 Einstein, in speaking of a conducting circuit and a magnet, claims that "the observable phenomenon here depends only on the relative motion of the conductor and the magnet."

Suppose that the situation is simplified by reducing the magnet to a single whirl which is located at the point **P** in the xy plane while the conducting circuit is replaced by a point charge q at the origin of co-ordinates (Fig. 4.1). When at rest the moment of the whirl is $\overline{k}m^0$ and the polarisation is zero.

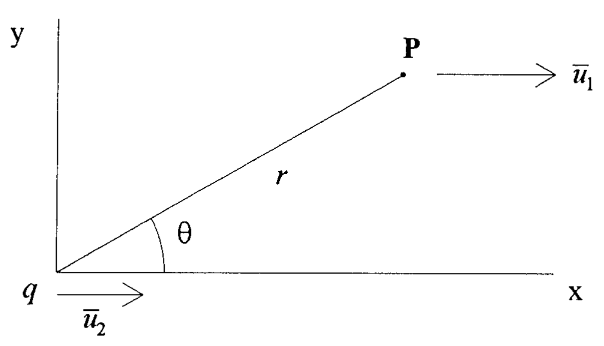


Fig. 4.1

If the whirl now translates with velocity $\bar{i}u_1$ and the charge with velocity $\bar{i}u_2$ show that the force acting upon the charge is given by

$$\bar{F} = \bar{j}q\left(u_2 - u_1\right)\frac{m^0}{cr^3}\frac{\left(1 - u_1^2/c^2\right)}{\left(1 - u_1^2\sin^2\theta/c^2\right)^{3/2}}$$

having first established that the presence of the moving charge does not affect the configuration adopted by the moving whirl.

It will be seen that the force cannot be expressed in terms of relative velocity, even when the movement is collinear, except for the case $u_1 = u_2$ when the force reduces to zero.

4.2 A doublet of moment $\bar{j}p$ translates in the xy plane with velocity $\bar{i}u$ (Fig. 4.2). The points **P** and **Q** are located in the xy plane as shown, and each is at a distance r from the doublet.

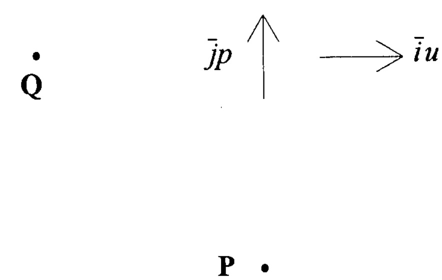


Fig. 4.2

Show that $\left(B_z\right)_P = \dfrac{2pu}{cr^3}\dfrac{1}{\left(1 - u^2/c^2\right)^{1/2}}$ and $\left(B_z\right)_Q = \dfrac{-pu}{cr^3}\left(1 - u^2/c^2\right)$

Suppose now that the doublet is replaced by an unpolarised whirl of moment $\bar{p} \times \dfrac{\bar{u}}{c}$.

Show that in this case

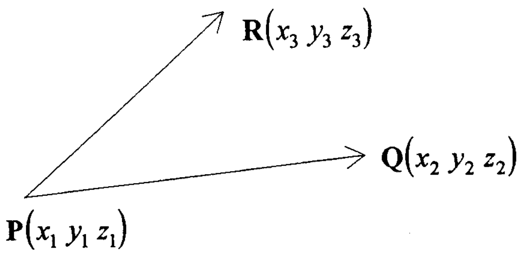
$$\left(B_z\right)_P = \frac{pu}{cr^3}\frac{\left(1 - 2u^2/c^2\right)}{\left(1 - u^2/c^2\right)^{3/2}} \quad \text{and} \quad \left(B_z\right)_Q = \frac{pu}{cr^3}\left(1 + u^2/c^2\right)$$

Observe that for small values of u/c the replacement results in $\left(B_z\right)_P$ being reduced to half its former value while $\left(B_z\right)_Q$ is reversed in sign.

APPENDICES

A.1 Transformation of Area

Let an elementary area be defined by the vectors $\overrightarrow{\mathbf{PQ}}$ and $\overrightarrow{\mathbf{PR}}$.



$$\overrightarrow{\mathbf{PQ}} = \bar{i}(x_2 - x_1) + \bar{j}(y_2 - y_1) + \bar{k}(z_2 - z_1)$$
$$\overrightarrow{\mathbf{PR}} = \bar{i}(x_3 - x_1) + \bar{j}(y_3 - y_1) + \bar{k}(z_3 - z_1)$$

The vector area of the parallelogram $\Delta\bar{S}$ is given by $\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}}$ i.e.

$$\begin{aligned}\Delta\bar{S} = \bar{i}&\{(y_2 - y_1)(z_3 - z_1) - (z_2 - z_1)(y_3 - y_1)\} \\ +\bar{j}&\{(z_2 - z_1)(x_3 - x_1) - (x_2 - x_1)(z_3 - z_1)\} \\ +\bar{k}&\{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\}\end{aligned}$$

If the elementary area translates in the aether frame with velocity components u_x, u_y, u_z, then, in a frame moving with velocity $\bar{i}v$ relative to the aether, we have[1]

$$\begin{aligned}\overrightarrow{\mathrm{P'Q'}} = \bar{i}\frac{(x_2 - x_1)}{\beta(1 - vu_x/c^2)} &+ \bar{j}\left\{y_2 - y_1 + \frac{vu_y}{c^2}\frac{(x_2 - x_1)}{(1 - vu_x/c^2)}\right\} \\ &+ \bar{k}\left\{z_2 - z_1 + \frac{vu_z}{c^2}\frac{(x_2 - x_1)}{(1 - vu_x/c^2)}\right\}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{P'R'}} = \bar{i}\frac{(x_3 - x_1)}{\beta(1 - vu_x/c^2)} &+ \bar{j}\left\{y_3 - y_1 + \frac{vu_y}{c^2}\frac{(x_3 - x_1)}{(1 - vu_x/c^2)}\right\} \\ &+ \bar{k}\left\{z_3 - z_1 + \frac{vu_z}{c^2}\frac{(x_3 - x_1)}{(1 - vu_x/c^2)}\right\}\end{aligned}$$

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1 F.A.P.T. Pt. 2, Sec. 1.7

Following vector multiplication and cancellation of terms, we obtain

$$\Delta\bar{S}' = \bar{i}\Bigg[(y_2 - y_1)(z_3 - z_1) - (z_2 - z_1)(y_3 - y_1) + \frac{vu_y}{c^2}\left\{\frac{(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)}{(1 - vu_x/c^2)}\right\} + \frac{vu_z}{c^2}\left\{\frac{(y_2 - y_1)(x_3 - x_1) - (x_2 - x_1)(y_3 - y_1)}{(1 - vu_x/c^2)}\right\}\Bigg]$$
$$+ \bar{j}\left\{\frac{(x_3 - x_1)(z_2 - z_1) - (x_2 - x_1)(z_3 - z_1)}{\beta(1 - vu_x/c^2)}\right\} + \bar{k}\left\{\frac{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)}{\beta(1 - vu_x/c^2)}\right\}$$

or

$$\Delta\bar{S}' = \bar{i}\left\{\Delta S_x - \frac{v/c^2}{(1 - vu_x/c^2)}\left(u_y\Delta S_y + u_z\Delta S_z\right)\right\} + \bar{j}\frac{\Delta S_y}{\beta(1 - vu_x/c^2)} + \bar{k}\frac{\Delta S_z}{\beta(1 - vu_x/c^2)} \tag{A.1-1}$$

from which we may derive the inverse relationship

$$\Delta\bar{S} = \bar{i}\left\{\Delta S'_x + \frac{v/c^2}{(1 + vu'_x/c^2)}\left(u'_y\Delta S'_y + u'_z\Delta S'_z\right)\right\} + \bar{j}\frac{\Delta S'_y}{\beta(1 + vu'_x/c^2)} + \bar{k}\frac{\Delta S'_z}{\beta(1 + vu'_x/c^2)} \tag{A.1-2}$$

If all elements of a surface translate with the same velocity, we may replace $\Delta\bar{S}$ by $\bar{S}$ and ΔS_x by S_x, etc.

A.2 Group Property of Lorentz Transformations for $\bar{\mathbf{E}}$, $\bar{\mathbf{B}}$, ρ, $\bar{\mathbf{J}}$, $\bar{\mathbf{M}}$ and $\bar{\mathbf{P}}$

Suppose that reference frames S' and S'' translate with velocities $\bar{i}v_1$ and $\bar{i}v_2$ with respect to the aether frame S.

Let $\beta_1 = (1 - v_1^2/c^2)^{-1/2}$ $\quad \beta_2 = (1 - v_2^2/c^2)^{-1/2}$ $\quad \beta_{12} = (1 - v_{12}^2/c^2)^{-1/2}$ (A.2-1)

where v_{12} is the velocity of S'' as measured in S'.

Expansion of v_{12} yields

$$\beta_{12} = \beta_1\beta_2\left(1 - v_1v_2/c^2\right) \tag{A.2-2}$$

The following relationships, which will be required later, are readily confirmed.

$$\left(1 + v_1u'_x/c^2\right) = 1/\beta_1^2\left(1 - v_1u_x/c^2\right) \tag{A.2-3}$$

$$\text{where} \quad u'_x = \left(u_x - v_1\right)/\left(1 - v_1u_x/c^2\right)$$

$$\left(1 - v_{12}u'_x/c^2\right) = \frac{\left(1 - v_2u_x/c^2\right)}{\left(1 - v_1u_x/c^2\right)}\frac{1}{\beta_1^2\left(1 - v_1v_2/c^2\right)} \tag{A.2-4}$$

$$\left(1 - v_2u_x/c^2\right)\left(1 + v_1u'_x/c^2\right) = \left(1 - v_1v_2/c^2\right)\left(1 - v_{12}u'_x/c^2\right) \tag{A.2-5}$$

(1) Transformation of $\overline{E}$ and $\overline{B}$

Since $E'_x = E_x$ and $E''_x = E_x$, $E''_x = E'_x$. Similarly $B''_x = B'_x$.

Of the remaining relationships we will consider only the transformation of E_y, since that of the other $\overline{E}$ and $\overline{B}$ components follow in a similar fashion.

Now

$$E'_y = \beta_1\left(E_y - \frac{v_1}{c}B_z\right) \qquad B'_z = \beta_1\left(B_z - \frac{v_1}{c}E_y\right)$$

$$E''_y = \beta_2\left(E_y - \frac{v_2}{c}B_z\right) \qquad B''_z = \beta_2\left(B_z - \frac{v_2}{c}E_y\right)$$

On elimination of E_y from (a), the left hand equations, and (b), the upper equations, we obtain in turn

$$E''_y = \beta_2\, E'_y/\beta_1 - \left(v_2 - v_1\right)\beta_2\, B_z/c \tag{A.2-6}$$

$$\text{and} \quad B_z = \beta_1\left(B'_z + \frac{v_1}{c}E'_y\right) \tag{A.2-7}$$

Then substitution of (A.2-7) in (A.2-6) yields

$$E''_y = \beta_1\beta_2\left(1 - v_1v_2/c^2\right)\left\{E'_y - \frac{\left(v_2 - v_1\right)/c}{\left(1 - v_1v_2/c^2\right)}B'_z\right\} = \beta_{12}\left(E'_y - \frac{v_{12}}{c}B'_z\right) \tag{A.2-8}$$

Thus the transformation between S' and S'' takes the same form as that between S and S'.

(2) Transformation of ρ and $\bar{\mathbf{J}}$

Since $J'_y = J_y$ and $J''_y = J_y$, $J''_y = J'_y$. Similarly $J''_z = J'_z$.

Also

$$J'_x = \beta_1\left(J_x - \rho v_1\right) \qquad \rho' = \beta_1\left(\rho - J_x v_1/c^2\right)$$

$$J''_x = \beta_2\left(J_x - \rho v_2\right) \qquad \rho'' = \beta_2\left(\rho - J_x v_2/c^2\right)$$

On elimination of J_x from (a), the left hand equations, and (b), the upper equations, we obtain in turn

$$J''_x = \beta_2 J'_x/\beta_1 - \left(v_2 - v_1\right)\beta_2\rho \tag{A.2-9}$$

$$\text{and} \quad \rho = \beta_1\left(\rho' + v_1 J'_x/c^2\right) \tag{A.2-10}$$

Then substitution of (A.2-10) in (A.2-9) yields

$$J''_x = \beta_{12}\left(J'_x - \rho' v_{12}\right) \tag{A.2-11}$$

In like manner, elimination of ρ from the upper and the right hand equations in turn, and subsequent combination, yields

$$\rho'' = \beta_{12}\left(\rho' - v_{12}J'_x/c^2\right) \tag{A.2-12}$$

(3) Transformation of $\overline{\mathbf{M}}$

We have
$$M'_x = M_x - \frac{v_1/c^2}{\left(1 - v_1 u_x/c^2\right)}\left(u_y M_y + u_z M_z\right)$$

and
$$M''_x = M_x - \frac{v_2/c^2}{\left(1 - v_2 u_x/c^2\right)}\left(u_y M_y + u_z M_z\right)$$

whence
$$M''_x = M'_x - \frac{\left(v_2 - v_1\right)\left(u_y M_y + u_z M_z\right)}{c^2\left(1 - v_1 u_x/c^2\right)\left(1 - v_2 u_x/c^2\right)} \tag{A.2-13}$$

But from (2.1-15), $M'_y = M_y/\beta_1\left(1 - v_1 u_x/c^2\right)$ etc., so that (A.2-13) may be brought into the form

$$M''_x = M'_x - \frac{\left(v_2 - v_1\right)\left(u'_y M'_y + u'_z M'_z\right)}{c^2\left(1 - v_2 u_x/c^2\right)\left(1 + v_1 u'_x/c^2\right)}$$

Then from (A.2-5)

$$M_x'' = M_x' - \frac{(v_2 - v_1)(u_y' M_y' + u_z' M_z')}{c^2(1 - v_1 v_2/c^2)(1 - v_{12} u_x'/c^2)}$$

or

$$M_x'' = M_x' - \frac{v_{12}/c^2}{(1 - v_{12} u_x'/c^2)}(u_y' M_y' + u_z' M_z') \tag{A.2-14}$$

We also have

$$M_y' = \frac{M_y}{\beta_1(1 - v_1 u_x/c^2)} \qquad \text{and} \qquad M_y'' = \frac{M_y}{\beta_2(1 - v_2 u_x/c^2)}$$

Then from (A.2-2) and (A.2-4)

$$M_y'' = M_y' \frac{\beta_1(1 - v_1 u_x/c^2)}{\beta_2(1 - v_2 u_x/c^2)} = \frac{M_y'}{\beta_{12}(1 - v_{12} u_x'/c^2)} \tag{A.2-15}$$

Similarly for M_z

(4) <u>Transformation of $\overline{\mathrm{P}}$</u>

Since $P_x' = P_x$ and $P_x'' = P_x$, $P_x'' = P_x'$

Also

$$P_y' = \beta_1(1 - v_1 u_x/c^2)P_y + \beta_1 v_1 u_y P_x/c^2 + \beta_1 v_1 M_z/c$$

and

$$P_y'' = \beta_2(1 - v_2 u_x/c^2)P_y + \beta_2 v_2 u_y P_x/c^2 + \beta_2 v_2 M_z/c$$

whence, by elimination of P_y,

$$P_y'' = \frac{\beta_2(1 - v_2 u_x/c^2)}{\beta_1(1 - v_1 u_x/c^2)}\left(P_y' - \beta_1 v_1 u_y P_x/c^2 - \beta_1 v_1 M_z/c\right) + \beta_2 v_2 u_y P_x/c^2 + \beta_2 v_2 M_z/c$$

or

$$P_y'' = \frac{\beta_2(1 - v_2 u_x/c^2)}{\beta_1(1 - v_1 u_x/c^2)}\left\{P_y' - \frac{v_1(u_y' P_x'/c^2 + M_z'/c)}{(1 + v_1 u_x'/c^2)}\right\} + \frac{\beta_2 v_2(u_y' P_x'/c^2 + M_z'/c)}{\beta_1(1 + v_1 u_x'/c^2)} \tag{A.2-16}$$

The coefficient of P_y' may be simplified by means of (A.2-4) and (A.2-2). We have

$$\frac{\beta_2(1 - v_2 u_x/c^2)}{\beta_1(1 - v_1 u_x/c^2)} = \beta_1\beta_2(1 - v_1 v_2/c^2)(1 - v_{12} u_x'/c^2) = \beta_{12}(1 - v_{12} u_x'/c^2)$$

The coefficient of P_x' reduces to

$$\frac{\beta_2(v_2 - v_1)\,u_y'/c^2}{\beta_1(1 + v_1u_x'/c^2)(1 - v_1u_x/c^2)}$$

$$= \beta_1\beta_2(v_2 - v_1)\,u_y'/c^2 \qquad \text{from (A.2 - 3)}$$

$$= \beta_1\beta_2(1 - v_1v_2/c^2)\frac{(v_2 - v_1)\,u_y'/c^2}{(1 - v_1v_2/c^2)}$$

$$= \beta_{12}v_{12}\,u_y'/c^2 \qquad \text{from (A.2 - 2)}$$

By the same procedure, the coefficient of M_z' is $\beta_{12}\,v_{12}/c$.

Hence

$$P_y'' = \beta_{12}(1 - v_{12}u_x'/c^2)P_y' + \beta_{12}v_{12}u_y'\,P_x'/c^2 + \beta_{12}v_{12}\,M_z'/c \qquad \text{(A.2-17)}$$

Similarly

$$P_z'' = \beta_{12}(1 - v_{12}u_x'/c^2)P_z' + \beta_{12}v_{12}u_z'\,P_x'/c^2 - \beta_{12}v_{12}\,M_y'/c \qquad \text{(A.2-18)}$$

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19. Abstract

An aether-based treatment of space and time measurement is employed to investigate the rate of a moving clock, to develop doppler formulae, and to establish intra and inter-frame relationships in electrodynamics.